MATH4210: Financial Mathematics Tutorial 6

WONG, Wing Hong

The Chinese University of Hong Kong

whwong@math.cuhk.edu.hk

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Internal Rate of Return

Question

Consider a 30-year \$2000 bond, that has coupons every 1/2 year in the amount of \$20, for a total of 60 times until 30 years at which time you receive \$2020. The bond price is \$2100. What is the yield (i.e. internal rate of return) if the rate is continuously compouning?

Answer

Let r be the internal rate of return. Then,

$$-2100 + \sum_{i=1}^{59} 20e^{-ir/2} + 2020e^{-30r} = 0$$

Internal Rate of Return

Answer

Write $x = e^{-r/2}$. *Then, we have*

$$f(x) := -2100 + \sum_{i=1}^{59} 20x^i + 2020x^{60} = -2100 + 20x(\frac{1 - x^{59}}{1 - x}) + 2020x^{60}$$
$$= 0$$
$$f'(x) = \frac{20 - 1200x^{59}}{1 - x} + \frac{20x - 20x^{60}}{(1 - x)^2} + 121200x^{59}.$$

Internal Rate of Return

Answer

To apply Newton's method solve $f(x) = 0$, *we use the initial guess* $x_0 = 0.99$, *then*

$$
\begin{aligned}
x_0 &= 0.99, \\
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 0.991195\ldots \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 0.991159\ldots
\end{aligned}
$$

Repeatedly, this yields the solution
$x^* \approx 0.991159, r = -2\log(x^*) = 0.017761$

Common mistake in Assignment 1: did not compute the value until x_n converges.

Options

Common mistakes in Assignment 2:

1. $S_0 = S_t e^{-rt}$ or short S_{t_D} cash at time $t < t_D$
2. Suppose a proportion d of the stock is paid at time t_D. Construct a portfolio: long a stock at time $t < t_D$, then the values of the portfolio are:

$$\Pi(t) = S(t)$$
$$\Pi(t_D) = (1 + d)S(t_D)$$
$$\Pi(T) = (1 + d)S(T)$$

3. Short one American call option and do not exercise it until it matures.

Fact:

1. You NEVER know the stock price in future.
2. S_{t_D} is a RANDOM variable, you cannot short a random cash!
3. You must REINVEST the money at time t_D to long d the stock in order to have:

$$\Pi(t) = S(t)$$
$$\Pi(t_D) = (1+d)S(t_D)$$
$$\Pi(T) = (1+d)S(T)$$

4. If you short one American call option, you have an OBLIGATION to exercise it upon request, i.e. you need to analyse what happens when the option is exercised at every time $t' \in (t, T]$

Options

Question

Suppose that we have the following 4 European call and put options with the same maturity T in the financial market:

Type	*Strike Price*	*Price*
Call	100	45
Call	110	40
Put	100	36
Put	110	42

Suppose that the continuous compounded interest rate is 5% in the market and the maturity time is $T = 1$, and assume the stock price is nonnegative. By considering the call-put parity, show that there is an arbitrage opportunity. Hence, or otherwise, construct a portfolio with arbitrage profit.

Options

Answer

Suppose not, i.e. there is no arbitrage opportunity. Recall the put-call parity that

$$C_E(t,K) - P_E(t,K) = S(t) - Ke^{-r(T-t)}.$$

In our case, we have

$$45 - 36 = S(0) - 100e^{-0.05} \quad (1)$$
$$40 - 42 = S(0) - 110e^{-0.05}. \quad (2)$$

(1) - (2): $11 = 10e^{-0.05}$, *which is a contradiction. Thus, there must be arbitrage opportunity.*

Answer

Now, we consider two portfolios: long one call option and short one put option with strike 100 *and* 110 *respectively at time* t*. Then, the values of the portfolios are:*

$$\Pi_1(t) = C_E(t, 100) - P_E(t, 100)$$
$$\Pi_2(t) = C_E(t, 110) - P_E(t, 110)$$

Now, we construct a new portfolio: long one Π_2 *and short one* Π_1 *and long* $10e^{-0.05}$ *cash.*

$$\Pi_3(t) = \Pi_2(t) - \Pi_1(t) + 10e^{-0.05} = 10e^{-0.05} - 11 < 0$$
$$\Pi_3(T) = \Pi_2(T) - \Pi_1(T) + 10 = S(T) - 110 + 100 - S(T) + 10 = 0,$$

which gives us an arbitrage profit.

Options

Remark: if you are asked to construct an arbitrage portfolio, unless other specification, it suffices to WRITE DOWN the portfolio and VERIFY $\Pi(0) \leq 0$, $\Pi(T) \geq 0$ and $\mathbb{P}(\Pi(T) > 0) > 0$ or any other equivalent definitions.

Question

Given American and European call options with strike K and maturity T. Assume that $S(T)$ has a lower bound $B \leq K$ and the underlying asset pays dividend D at time $t_D \in (t, T]$. Assume further that $D > K - Be^{-r(T-t_D)}$. Prove

$$C_A(t) > C_E(t)$$

for all $t < T$.

Answer

We first construct a portfolio by simply longing one European call option. Then, the values of the portfolio are

$$\Pi_1(t) = C_E(t)$$
$$\Pi_1(T) = (S(T) - K)^+.$$

Answer

Next, we construct another portfolio by longing one American call option and exercise it at time $t_1 < t_D$, where $t_1 > t$ is chosen to satisfy

$$D > Ke^{r(t_D - t_1)} - Be^{-r(T - t_D)}$$

Then, the values of the portfolio are

$$\begin{aligned}
\Pi_2(t) &= C_A(t) \\
\Pi_2(t_1) &= S(t_1) - K \\
\Pi_2(t_D) &= S(t_D) + D - Ke^{r(t_D - t_1)} \\
\Pi_2(T) &= S(T) + De^{r(T - t_D)} - Ke^{r(T - t_1)} \\
&> B + Ke^{r(T - t_1)} - B - Ke^{r(T - t_1)} \\
&= 0
\end{aligned}$$

Answer

Also, we have

$$\begin{aligned}\Pi_2(T) &= S(T) + De^{r(T-t_D)} - Ke^{r(T-t_1)} \\ &= S(T) - K + De^{r(T-t_D)} - K(e^{r(T-t_1)} - 1) \\ &> S(T) - K + Ke^{r(T-t_D)} - B - K(e^{r(T-t_1)} - 1) \\ &\geq S(T) - K + Ke^{r(T-t_D)} - K - K(e^{r(T-t_1)} - 1) \\ &> S(T) - K\end{aligned}$$

Hence,

$$\Pi_2(T) > (S(T) - K)^+ = \Pi_1(T).$$

Thus, we must have $\Pi_2(t) > \Pi_1(t)$, *i.e.* $C_A(t) > C_E(t)$.

Exercise:

Question

Suppose the continuous compounding interest rate is r and the price of a stock is $S(t)$ at time t. If it pays dividend $d \times S(t_D)$ at time $t_D \in (t, T]$ with $0 < d < 1$. Let $C_E(t, K)$ and $P_E(t, K)$ be the prices of European call and put option at time t with strike K and maturity T respectively. Show that

$$C_E(t, K) - P_E(t, K) = \frac{1}{1+d} S(t) - Ke^{-r(T-t)}$$

for all $t < T$.

Options

Exercise:

Question

Under the same setting, suppose two European put options has same strike K and different maturity $T_1 < T_2$, show that

$$P_E(t, T_1) < (1+d)P_E(t, T_2) + K(e^{-r(T_1-t)} - (1+d)e^{-r(T_2-t)})^+$$

Question

Deduce similar inequality for European call options.

Binomial Tree Models

Consider a two step binomial tree model with the following parameters: $t_k = k\Delta t$ for $k = 0, 1, 2$. $S_{t_0} = 100, u = 1.5, d = 0.5, e^{r\Delta t} = 1.1$.

Question

Compute the risk neutral probability measure and then the price as well as the replication strategy of the European put with strike $K = 100$ and maturity $T = t_2$.

Answer

$$q = \frac{e^{r\Delta t} - d}{u - d} = 0.6.$$

The risk neutral probability measure $\mathbb{Q}$ is given by

$$\begin{cases} \mathbb{Q}[S_{t_1} = uS_0] & = \mathbb{Q}[f(t_1) = f_u] = 0.6 \\ \mathbb{Q}[S_{t_1} = dS_0] & = \mathbb{Q}[f(t_1) = f_d] = 0.4 \end{cases}$$

Binomial Tree Models

Answer

The price of the option can be computed via

1. $S_0 = 100,\ S_u = 150,\ S_d = 50, S_{uu} = 225, S_{ud} = S_{du} = 75, S_{dd} = 25.$
2. $f_{uu} = 125, f_{ud} = f_{du} = f_{dd} = 0$
3. $f_u = e^{-r\Delta t}(qf_{uu} + (1-q)f_{ud}) = 68.18$
4. $f_d = e^{-r\Delta t}(qf_{du} + (1-q)f_{dd}) = 0$
5. $f = e^{-r\Delta t}(qf_u + (1-q)f_d) = 37.19$

Answer

The dynamic trading strategy is

1. $\phi_{t_1}^{u} = \frac{f_{uu} - f_{ud}}{S_{uu} - S_{ud}} = 0.8333$
2. $\phi_{t_1}^{d} = \frac{f_{du} - f_{dd}}{S_{du} - S_{dd}} = 0$
3. $\phi_{t_0} = \frac{f_u - f_d}{S_u - S_d} = 0.6818$

Question

Compute the price as well as the replication strategy of the American call with strike $K = 100$ and maturity $T = t_2$. Is its price higher than the European call?

Answer

The price of the option can be computed via

1. $S_0 = 100,\ S_u = 150,\ S_d = 50, S_{uu} = 225, S_{ud} = S_{du} = 75, S_{dd} = 25.$
2. $f_{uu} = 125, f_{ud} = f_{du} = f_{dd} = 0$
3. $f_u = \max(e^{-r\Delta t}(qf_{uu} + (1-q)f_{ud}), 150 - 100) = 68.18$
4. $f_d = \max(e^{-r\Delta t}(qf_{du} + (1-q)f_{dd}), 50 - 100) = 0$
5. $f = \max(e^{-r\Delta t}(qf_u + (1-q)f_d), 100 - 100) = 37.19$

Obviously. the trading strategy is the same as above, and its price is equal to European call.

Binomial Tree Models

Question

Compute the price of the European and American puts with strike $K = 100$ *and maturity* $T = t_2$. *Is American put's price higher than the European pus?*

Binomial Tree Models

Answer

The price of the European option can be computed via

1. $S_0 = 100,\ S_u = 150,\ S_d = 50, S_{uu} = 225, S_{ud} = S_{du} = 75, S_{dd} = 25.$
2. $f_{uu} = 0, f_{ud} = f_{du} = 25, f_{dd} = 75$
3. $f_u = e^{-r\Delta t}(qf_{uu} + (1-q)f_{ud}) = 9.09$
4. $f_d = e^{-r\Delta t}(qf_{du} + (1-q)f_{dd}) = 40.91$
5. $f = e^{-r\Delta t}(qf_u + (1-q)f_d) = 19.83$

Answer

The price of the American option can be computed via

1. $S_0 = 100,\ S_u = 150,\ S_d = 50, S_{uu} = 225, S_{ud} = S_{du} = 75, S_{dd} = 25.$
2. $f_{uu} = 0, f_{ud} = f_{du} = 25, f_{dd} = 75$
3. $f_u = \max(e^{-r\Delta t}(qf_{uu} + (1-q)f_{ud}), 100 - 150) = 9.09$
4. $f_d = \max(e^{-r\Delta t}(qf_{du} + (1-q)f_{dd}), 100 - 50) = 50$
5. $f = \max(e^{-r\Delta t}(qf_u + (1-q)f_d), 100 - 100) = 23.14$

The American put has a higher value than the European put.