MATH4210: Financial Mathematics Tutorial 2

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23 September, 2020

Bond

Recall the following definition. Let P be the price of a bond with cash flows c_i at time $t_i, i = 1, 2, ..., n$ and λ be the bond yield, then

$$P = \sum_{i=1}^{n} c_i e^{-\lambda t_i}.$$

The bond duration D is defined by

$$D := -\frac{1}{P}\frac{\partial P}{\partial \lambda} = \frac{1}{P}\sum_{i=1}^{n} t_i c_i e^{-\lambda t_i},$$

which measures the sensitivity of the bond price with respect to small changes in the yield.

Bond

The bond duration C is defined by

$$C := \frac{1}{P}\frac{\partial^2 P}{\partial \lambda^2} = \frac{1}{P}\sum_{i=1}^{n} t_i^2 c_i e^{-\lambda t_i},$$

Bond

Question

Consider a semiannual coupon bond mature in 12 months. Assume that the face value of the bond is \$1000 *and has coupons every* $1/2$ *year in the amount of* \$10,*. If the bond price is* \$1000, *find the bond yield.*

Answer

- *Let* λ *be the bond yield.*
- $t_1 = 1/2, t_2 = 1$
- $c_1 = 10, c_2 = 1010$
- *Let* $x = e^{-\lambda/2}$, *by considering the net present value (NPV) we have* $1000 - 10x - 1010x^2 = 0$.
- *Solve it, we have* $x = 100/101$, $\lambda = 0.0199$.

Bond

Question

Consider a semiannual coupon bond with coupon rate 5% and mature in 24 months. Assume that the face value of the bond is $100, and interest is compounded continuously. Compute the price of the bond if the zero rate is

$$r(0,t) = 0.05 + \frac{\ln(1+t)}{100},$$

i.e. the rate of return of a cash deposit made at time 0 and maturing at time t. Find the bond price.

Bond

Answer

- $t_1 = 1/2$, $t_2 = 1$, $t_3 = 3/2$, $t_4 = 2$
- $c_1 = 2.5$, $c_2 = 2.5$, $c_3 = 2.5$, $c_4 = 102.5$
- $e^{-r(0,t_1)t_1} = 0.973$, $e^{-r(0,t_2)t_2} = 0.945$, $e^{-r(0,t_3)t_3} = 0.915$, $e^{-r(0,t_4)t_4} = 0.885$
- $P = \sum_{i=1}^{n} c_i e^{-r(0,t_i)t_i} = 97.8$

Bond

Question

Consider a semiannual coupon bond with coupon rate 5% *and mature in* 24 *months. If the bond yield is* 6% *and the face value is* \$100, *compute the price, the duration and the convexity of the bond.*

Answer

- $t_1 = 1/2, t_2 = 1, t_3 = 3/2, t_4 = 2$
- $c_1 = 2.5, c_2 = 2.5, c_3 = 2.5, c_4 = 102.5$
- $\lambda = 0.06$
- $P = \sum_{i=1}^{4} c_i e^{-rt_i} = 97.975$
- $D = \sum_{i=1}^{4} t_i c_i e^{-rt_i} / P = 1.927$
- $C = \sum_{i=1}^{4} t_i^2 c_i e^{-rt_i} / P = 3.794$

Swap Contract

Question

Suppose the continuous interest rate of Chinese Yuan (CNY) and Hong Kong Dollar (HKD) are 3% and 2% respectively and the spot exchange rate is HKD 1.1/ CNY. Your company is expected to receive CNY 20000 for the next 5 years and wants to swap this for a fixed HKD obligation of amount HKD x. What is x?

Answer

By considering the present value, we have

$$
\begin{aligned}
NPV(HKD) &= \text{exchange rate} \times NPV(CNY) \\
x(e^{-0.02} + e^{-0.04} + \cdots + e^{-0.1}) &= 1.1 \times 20000(e^{-0.03} + \cdots + e^{-0.15}) \\
x &= 21360.47
\end{aligned}
$$

Arbitrage Opportunity

Recall that an arbitrage opportunity is a trading opportunity that either (1) takes a negative amount of cash to enter (i.e. cash flow to your pocket is positive) and promises a non-negative payoff to leave; or (2) takes a non-positive amount of cash to enter (cash flow to you is either zero or positive) and promises a non-negative payoff with the possibility of a positive payoff when leaving.
Let $\Pi(t)$ be the portfolio at time t. Mathematically speaking, we said

Definition

There is an arbitrage opportunity if any one of the following holds:
(1) $\Pi(t_1) < 0$ and $\Pi(t_2) \geq 0$ for some $t_2 > t_1$; or
(2) $\Pi(t_1) \leq 0$ and $\Pi(t_2) \geq 0$ with $\mathbb{P}(\Pi(t_2) > 0) > 0$ for some $t_2 > t_1$

Roughly speaking, arbitrage opportunity is the *possibility to earn money without risk*.

Arbitrage Opportunity

Question

Show that the above definition of arbitrage opportunity is equivalent to the existence of some $t_2 > t_1$ such that $\Pi(t_1) = 0$ and $\Pi(t_2) \geq 0$ with $\mathbb{P}(\Pi(t_2) > 0) > 0$.

Answer

(<=) part is trivial. Let's prove the (=>) part. Assume (1) is true in the definition, i.e. there exists some $t_2 > t_1$ such that $\Pi(t_1) < 0$ and $\Pi(t_2) \geq 0$. We can withdraw \$$(-\Pi(t_1))$ from the portfolio at time t_1 and put it into the portfolio at time t_2. Our new portfolio $\tilde{\Pi}$ at time t_1 and t_2 are

$$\tilde{\Pi}(t_1) = \Pi(t_1) - \Pi(t_1) = 0$$

$$\tilde{\Pi}(t_2) = \Pi(t_2) - \Pi(t_1) > \Pi(t_2) \geq 0$$

In particular, $\tilde{\Pi}(t_2) \geq 0$ and $\mathbb{P}(\tilde{\Pi}(t_2) > 0) = 1 > 0$. The remaining parts are similar.

Stock

Question

Suppose there is no arbitrage opportunity. Let $S(t)$ be the price for some stock. Assume that $S(t) \geq 0$ for all t. Show that if $S(t_0) = 0$ for some t_0, then $S(t) = 0$ for all $t \geq t_0$.

Answer

Suppose not. There exists some $t_1 > t_0$ such that $S(t_1) > 0$. We buy 1 stock at $t = t_0$ and sell 1 stock at $t = t_1$. Then, the value of the portfolio is

$$\Pi(t_0) = 0$$
$$\Pi(t_1) = S(t_1) > 0,$$

which contradicts the (2) in our definition of arbitrage opportunity. Thus, we must have $S(t) = 0$ for all $t \geq t_0$.

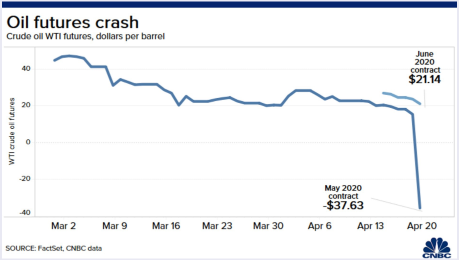
Stock

Question

Why we need to assume $S(t) \geq 0$ during our proof?

Answer

The value of some derivatives can be NEGATIVE in reality.



If the price has possibility to go downwards to negative infinity, there is always risks, i.e. no arbitrage opportunity. It becomes inconclusive in this case.

Forward

Forward Contract: Eliminate the risk for both parties. At current time t_0,

the price of a certain product is $p(t_0)$. It is possible that at time $t_1 > t_0$, the price of the product can raise up to $p(t_1)$, which is greater that $p(t_0)$. Therefore, the buyer buys a contract from the seller, promising that he will buy the product at price p at time t_1, where $p(t_0) < p < p(T)$. The buyer eliminates the risk that the price of the product may raise over p, while the seller eliminates the risk that the price of the product may not raise over p.

Future

Futures Contract: Control the risk within the contract period. At current

time t_0, the buyer and the seller deposit the same amount of money M into a broker's account. Suppose the maintenance level is set to be $m < M$. At the end of each day t beyond t_0, the price of the product is calculated. If the price increases (decreases) from the previous day, the seller (buyer) has to pay $x(t)$ dollars to the the buyer (seller) from his own account. If either account falls below the maintenance level, the account owner will be asked to invest money into the account. If the account does invest, the contract is automatically continued. Otherwise, the contract automatically ends. The remaining deposit money is given back to both parties correspondingly. The maintenance level m bounds the risk for both parties.

Options

One party buys the RIGHT to buy or sell in the future from the other party. There are mainly two types of options:

- Call option: Holder has the right to buy (long call position); Writer has the obligation to sell (short call position).
- Put option: Holder has the right to sell (long put position); Writer has the obligation to buy (short put position).

There are different kind of options according to the policy in different places:

- European option: Can be exercised at maturity date only
- American option: Can be exercised at any day before the maturity date.
- Asian option, Barrier option, etc.

Forward & Options

Question

Suppose the continuous compounded interest rate is r. Mr. Chan went to a stand-up comedy show at time recently. He loves the show and he wants to attend to the performer's next show, which is very likely to be held next year. The price of the ticket this year is \$400*. Mr. Chan thinks that the price of the ticket for the show next year may raise up to* \$600*. We compare two cases: (1) Mr. Chan signs a forward contract with the performer at* \$500 *for the ticket next year; (2) Mr. Chan signs a call option with the performer at* \$50*, giving the right Mr. Chan to buy the ticket at* \$500 *next year. Analyze the portfolio in both cases and find the maximum amount of money that Mr. Chan can lose.*

Answer

For case (1), Mr. Chan pays nothing at $t = 0$, so

$$\Pi_1(0) = 0.$$

Now, at $t = 1$, both Mr. Chan and the performer must obey the obligation, i.e. Mr. Chan must pay \$500 to buy the ticket. Thus,

$$\Pi_1(1) = P - 500,$$

where \$$P$ is the price of the ticket at $t = 1$.
(What if somebody does not follow the obligation?)

Answer

For case (2), Mr. Chan pays \$50 *at* $t = 0$*, which is equivalent to borrow* \$50 *from the bank and buy the options, so*

$$\Pi_2(0) = 50 - 50 = 0.$$

Now, at $t = 1$*, Mr. Chan will exercise the ticket only if the price is higher than* \$500*, otherwise he can buy the ticket from market. Also, he must pay back the bank* $\$50e^r$*. Thus,*

$$\Pi_2(1) = \max(P - 500, 0) - 50e^r = (P - 500)^+ - 50e^r,$$

where $\$P$ *is the price of the ticket at* $t = 1$*.*

Forward & Options

Answer

Since we always have $P \geq 0$ (why?) and there is a possibility that the ticket price becomes 0, thus, the maximum amount of money that Mr. Chan can lose is

$$\min_{P\geq 0} \Pi_1(1) - \Pi_1(0) = \min_{P\geq 0} P - 500 = -500$$

for case (1) and

$$\min_{P\geq 0} \Pi_2(1) - \Pi_2(0) = \min_{P\geq 0} (P - 500)^+ - 50e^r = -50e^r$$

for case (2).

Why $P \geq 0$?

Answer

Suppose not, i.e. $P < 0$. *We can buy the ticket, throw it into a rubbish bin and earn* \$$(-P)$. *This is an arbitrage opportunity.*