MATH4210: Financial Mathematics

II. Interest Rate

Interest Type

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- Simple interest: Which is based only on the amount of initially invest (principle).
- Compound interest: Additional to simple interest, also get interest on the interest
  - Discrete compound interest: The process of discrete compounding is utilized at specific finite periods of time, such as daily, monthly, or annually.
  - Continuous compound interest: Calculating the compounding period infinitesimally small.

Simple Interest

Suppose that you place $\$x_0$ in an account in a bank that offers a fixed (never to change over time) annual simple interest rate of $r > 0$ (per year).

– $\$ x_0$ is called the principal, and one year later at time $t = 1$ you will have the amount

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- Essentially, at the end of the investment period you collect an amount of money which is the sum of the principal (the original amount you owned), plus interest.

Simple Interest

In other words, assume you have wealth $\$x_0$, and you invest it in, say, a bank account for one year. After the one year period, you will get an amount of money $\$y_1 > \$x_0$. Hence, you could measure the annual simple interest rate of return of your investment by

$$r = \frac{y_1 - x_0}{x_0}.$$

If the period is n year, and your final collected money is $\$y_n$, then the simple interest rate is computed by

$$r = \frac{1}{n}\frac{y_n - x_0}{x_0}.$$

Zero-coupon Bond

Example 1.1

A 2-year zero-coupon bond has a face value $\$100$.

What is the value of it if the annual simple interest rate is 5%*?*

What is the (implied) annual simple interest rate if the bond is issued for $\$80$*?*

Zero-coupon Bond

Discussion

The bond price should be

$$100(1 + 5\% \times 2)^{-1} = 90.9,$$

when the annual simple interest rate is 5%.

Suppose the (implied) annual simple interest rate is r when the bond is issued for $\$80$. Then

$$100(1 + 2r)^{-1} = 80,$$

and therefore, $r = 12.5\%$.

Interest rate

Everything here is deterministic, in that the amount y_1 is not random; there is no uncertainty — hence no risk — associated with this investment. This is an example of a risk-free investment.

Remark 1

Investing in stocks on the other hand is an example of a risky investment. This is because the price of stocks evolves over time in a random way, it is not possible to predict with certainty. The stock price at a later time thus has a non-zero variance. As with gambling, when investing in stock there is even a risk of losing some or all of your principal.

Discrete Compound Interest

If you reinvest y_1 for another year, then at time $t = 2$ you would have the amount

$$y_2 = y_1(1 + r) = x_0(1 + r)^2.$$

What if you keep doing this year after year ?

$$\begin{aligned}
\text{1st}: &\quad y_1 = x_0(1 + r) \\
\text{2nd}: &\quad y_2 = x_0(1 + r)^2 \\
\text{3rd}: &\quad y_3 = x_0(1 + r)^3 \\
\text{4th}: &\quad y_4 = x_0(1 + r)^4 \\
&\quad \ldots
\end{aligned}$$

In general, if you keep doing this year after year, then at time $t = n$ years, your payoff would be

$$y_n = x_0(1 + r)^n.$$

Discrete Compound Interest

Reinvesting the payoff at the end of one time period for yet another is known as compounding the interest.

In our example interest was compounded annually, but compounding could be done semiannually (or biannually), quarterly, monthly, daily and so on.

Discrete Compound Interest

Suppose the interest is payed every half-year with rate $r/2$. Then the value of the invest after half a year will be

$$x_0(1 + r/2).$$

Now we deposit $x_0(1 + r/2)$ (new principal) in the bank account again. The another half-year later, the value of the invest becomes

$$x_0(1 + r/2)(1 + r/2) = x_0(1 + r/2)^2.$$

If it is compounded m times per annum, the terminal value of the investment after one year is

$$x_0(1 + r/m) \cdots (1 + r/m) = x_0(1 + r/m)^m.$$

After n years, it is

$$x_0(1 + r/m)^m \cdots (1 + r/m)^m = x_0(1 + r/m)^{nm}.$$

Discrete Compound Interest

Example 1.2

When the interest is payed every half-year, at time $t = 0.5$, we would earn interest at the rate $0.5r > 0$ yielding $x0(1 + 0.5r)$, then half a year later at time $t = 1$ interest would be earned again at the rate $0.5r$ on the amount $x_0(1+0.5r)$ finally yielding a payoff at time $t = 1$ of the amount:

$$\begin{aligned} y_1 &= x_0(1 + 0.5r)(1 + 0.5r) \\ &= x_0(1 + 0.25r^2 + r) \\ &> x_0(1 + r). \end{aligned}$$

Thus, as expected, semiannual compounding is more profitable than annual compounding, As is easily seen, the profit increases as the time period for compounding get smaller.

Continuous Compound Interest

Over an arbitrary interval of time, $(0, t]$, compounding a total of k times would mean compounding every $\frac{t}{k}$ units of time at the rate $\frac{rt}{k}$ and yielding a payoff at time t of

$$y_t(k) = x_0 \left(1 + \frac{rt}{k}\right)^k .$$

Let the compounding interval get smaller and smaller, or equivalently, let $k \to \infty$, in which case the compounding becomes continuous compounding yielding a payoff at time t of

$$\lim_{k\to\infty} y_t(k) \;=\; \lim_{k\to\infty} x_0 \left(1 + \frac{rt}{k}\right)^k \;=\; x_0 e^{rt}.$$

Continuous Compound Interest

Remark 2

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

n (Compounding frequency)	$(1+1/n)^n$ (value of \$1 in one year)
1	2
2	2.25
4	2.44141
12	2.61304
52	2.66373
365	2.69260
10000	2.71815
1000000	2.71828

Continuous Compound Interest

Remark 3

An initial investment of x_0 at time $t = 0$, under continuous compound interest at rate r, is worth $x_0 e^{rt}$ at time $t \geq 0$.

Doubling your money

Example 1.3

[Question]: If the annual interest rate is r*, and you invest* x_0 *under continuous compound interest, then how long will you have to wait until you have doubled your money?*

[Solution]: We wish to find the value of t *(years) for which* $xe^{rt} = 2x$ *, or or equivalently, for which* $e^{rt} = 2$*.*

Solving this we conclude

$$t = \frac{\log(2)}{r} = \frac{0.6931}{r}.$$

If $r = 0.04$*, then* $t \approx 17$ *years, i.e.,*

$$t = \frac{\log(2)}{0.04} \approx 17.$$

Effective and Nominal interest rates

– In the discrete compound Interest model, where the interest is compounded m times per annum, the terminal value of the investment after one year is $x_0(1+r/m)^m$,

r is called the (annual) nominal interest rate.

Nominal rate does not reflect the "true interest rate"!

What is the " true interest rate"? The true simple interest rate of the year is computed by

$$1+r^* = \frac{x_0(1+r/m)^m}{x_0} = (1+r/m)^m.$$

We call r^* the the (annual) effective interest rate.

$$\text{effective rate} = \left(1 + \frac{\text{nominal rate}}{m}\right)^m - 1.$$

Effective and Nominal interest rates

Example 1.4

If compounding is done semiannually, then the effective rate is the solution r^* *to the equation*

$$1 + r^* = (1 + 0.5r)^2,$$

which yields

$$r^* = r + 0.25r^2.$$

So if $r = 0.07$*, then*

$$r^* = 0.0712.$$

Effective and Nominal interest rates

Which one is bigger ? Why?

Effective and Nominal interest rates

– In the continuous compound Interest model with compound interest rate r, the (annual) nominal interest rate is r, the terminal value of the investment after one year is $x_0 e^r$ the (annual) effective interest rate is

$$r^* = e^r - 1.$$

Example 1.5

If $r = 0.07$ and the compounding is monthly, then

$$r^* = \left(1 + \frac{0.07}{12}\right)^{12} - 1 = 0.0723.$$

If compounding is continuous, then

$$r^* = e^{0.07} - 1 = 0.0725.$$

Present value

Any given amount of money, x_0, is worth more to us at present time $t = 0$ than being promised it at some time $t > 0$ in the future, since we can always use x_0 now as principal in a risk-free investment at (continuous compound interest) rate $r > 0$ guaranteeing the amount

$$x_0 e^{rt} > x_0$$

at time t.

Present value

- Stated differently, an amount x promised to us later at time t is only worth

$$x_0 = xe^{-rt} \; < \; x$$

at present, since using this x_0 as principal yields exactly the amount x at time t:

$$x_0 e^{rt} = xe^{-rt} e^{rt} = x.$$

We thus call

xe^{-rt} the present value (PV) of x.

It is less than x and represents the value to us now of being promised x at a later time t in the future, if the interest rate is $r > 0$.

- We also say xe^{-rt} is the discounted value of x at the future time t, and the factor e^{-rt} is called the discount factor.

Present value

- In the discrete compound interest rate model, where one compounds k times during $(0, t]$ at annual rate r,

$$\text{the present value (PV) of } x \text{ at time } t \text{ is } x(1 + rt/k)^{-k}.$$

The discount factor is

$$(1 + rt/k)^{-k}.$$

Present value

[Question]: You are to receive $10,000$ one year from now and another $10,000$ two years from now. If the annual nominal interest rate is $r = 0.05$, compounded yearly, what is the present value of the total? What if compounded monthly? Continuously?

- When compounded yearly, then

$$PV = 10,000 * (1 + r)^{-1} + 10,000 * (1 + r)^{-2}.$$

- When compounded monthly, then

$$PV = 10,000 * (1 + r/12)^{-12} + 10,000 * (1 + r/12)^{-2*12}.$$

- When compounded continuously, then

$$PV = 10,000 * e^{-r} + 10,000 * e^{-2r}.$$

Plugging in $r = 0.05$ yields

$$PV = 18,561.$$

Present value

[Question]

- Suppose today is January 1st, and you move into a new apartment in which you must pay a monthly rent of $1000 on the first of every month, starting January 1st.
- You plan to move out after exactly one year. Then at the end of one year you will have paid a total of $12, 000.
- But suppose that you want to pay off all this 12 month future debt now on January 1st
- How much should you pay assuming that the nominal annual interest rate is $r = 0.06$ and interest will be compounded monthly?
- How much if the interest is compounded continuously ?

Net Present Value

[Question]:

- You win a very special lottery which agrees to pay you $100 each year forever starting today.
- Even after you die, the payments will continue to your relatives/friends.
- Assuming an interest rate of $r = 0.08$ compounded annually (and assumed constant forever), what is the present value of this lottery?
- What if compounded continuously?

Net Present Value

- Assessing the present value of an investment, for purposes of comparison with another investment, involves allowing for negative amounts (money spent).
- In this case we call the present value the net present value (NPV).
- More precisely, let x_i denote the money earned/spent at year i yields a vector $(x_0, x_1, x_2, \cdots, x_n)$ representing the investment up to time $t = n$, and its NPV (when interest is compounded yearly at rate r) is given by

$$\text{NPV} = \sum_{i=0}^{n} x_i (1+r)^{-i}.$$

Remark 4

The investment with the largest (positive) NPV is the better investment using this criterion.

Net Present Value

Example 2.1

Let $(-5, 1, 3, 0, -1)$ *be an investment which represents that*

- *one spends* $\$5$ *at time* 0.
- *one earns* $\$1$ *at* $t = 1$,
- *earns* $\$3$ *at* $t = 2$,
- *earns nothing at time* $t = 3$,
- *spent* $\$1$ *a time* $t = 4$.

Then the NPV is computed by

$$-5\times(1+r)^0+1\times(1+r)^{-1}+3\times(1+r)^{-2}+0\times(1+r)^{-3}+(-1)\times(1+r)^{-4}.$$

Loan formula

[Question]: Suppose you wish to take out a loan for $\$P$, at monthly interest rate r requiring you to make payments each month (starting 1 month from now), for n months. How much must you pay per month?
[Solution]: Let A be the monthly payment, then

$$P = \sum_{i=1}^{n} A(1+r)^{-i},$$

which implies that

$$A = P\frac{r(1+r)^n}{(1+r)^n - 1}.$$

Remark 5

When $n \to \infty$ (the perpetual situation), one obtains that

$$A = \lim_{n\to\infty} P\frac{r(1+r)^n}{(1+r)^n - 1} = rP.$$

Loan formula

[Question]:

- Jennifer has a 15 year home mortgage.
- She needs to pay \$2300 at the end of each month for the next 15 years.
- The interest on the loan is compounded monthly with annual nominal rate 3.625%.
- She is having trouble affording the \$2300 per month.
- To lower her monthly payment, she is going to refinance to a 30 year loan which has annual nominal rate 4.5% compounded monthly.
- What is her new monthly payment ?

NPV and IRR

Given a deterministic cash flow steam, $(x_0, x_1, \cdots, x_n)$, where x_i (allowed to be positive, 0 or negative) denotes the flow at time period i (say years), then the Net Present Value (NPV) is given by

$$NPV := \sum_{i=0}^{n} x_i(1+r)^{-i},$$

given the (annual) compound interest rate r.

Given a cash flow stream $(x_0, x_1, \cdots, x_n)$, the associated Internal Rate of Return (IRR) r_{irr} is the rate under which the corresponding NPV equals to 0, i.e. r_{irr} is the solution of the equation

$$\sum_{i=0}^{n} x_i(1+r_{irr})^{-i} = 0.$$

IRR

Theorem 1

Assume that $x_0 < 0$ *and* $x_i \geq 0$ *for all* $i = 1, \cdots, n$*, and* $\sum_{i=1}^{n} x_i > 0$*. Then IRR equation has a solution* $r_{irr} > -1$*.*

IRR

- IRR can be used as an alternative to NPV for purposes of comparing two different streams to decide which is better.
- The idea is that of the two, you would choose the one having the largest IRR. Of course, just as any stream with $NPV < 0$ would be avoided, any stream with $IRR < r$, where r is the current interest rate would be avoided too.
- Unfortunately it is possible that the two methods yield different conclusions; that is, IRR might rank your first steam higher than the second, while NPV might rank your second steam higher than your first!
- Finally we point out that a solution to IRR must be solved for numerically in general, using Newton's method, for example.

Bonds

A Bond is an example of a fixed income security, meaning that the payoff is essentially predetermined, deterministic, fixed.

- For a bound, one invests a fixed amount of money now and is guaranteed fixed, known payoffs in the future
- On the other hand, the stock is a so-called risky security, whose payoff is random and potentially highly volatile.

Zero Coupon Bond

The zero-coupon bond is a bond which involves just a single payment. The issuing institution (for example, a government, a bank or a company) promises to exchange the bond for a certain amount of money F , called the face value, on a given day T , called the maturity date.

- Given the interest rate, one can compute the present value (price) P of the zero coupon, i.e.

$$P = Fe^{-rT} \quad \text{(resp. } \; P = F(1+r)^{-T}),$$

  if r is the continuous compound (resp. annual compound) interest rate.

- During the lifetime of your bond before maturity, interest rates might change causing the change of the price of the bonds: if the rate goes up, the bond price goes down; if the rate goes down, the bond price goes up.

Zero Coupon Bond

- On the other hand, given the present value (price) P of the zero coupon bond, one can compute the interest rate:

$$r = \frac{1}{T}\log(F/P) \quad \text{or} \quad r = (F/P)^{1/T} - 1,$$

  depending on the fact that r is the continuous or discrete compound interest rate.
- In practice, the interest rate r (computed from the market price P) depends on the maturity T.

Coupon Bonds

The coupon bond is a financial asset promising a sequence of payments. These payments consist of the face value due at maturity, and coupons paid regularly, typically annually, semi-annually, or quarterly, the last coupon due at maturity.

Example 3.1

A coupon bond with maturity T, face value F and annual coupon K has the payment stream

$$(x_1 = K,\ x_2 = K, \cdots, x_{T-1} = K,\ x_T = K + F).$$

Given the interest rates, one can compute the present value (price) of a coupon bond by discounting all the future payments.

Yield of the bond

For a coupon bonds with price P, one defines the cash flow stream $(x_0, x_1, \cdots, x_n)$ induced by the bond by

$$x_0 = -P < 0, \quad x_i = \text{the payment at time } t_i.$$

The yield of the bond λ is defined as the IRR (internal rate of return) of the stream $(x_0, \cdots, x_n)$, i.e. the solution of the equation

$$\sum_{i=0}^{n} x_i (1+\lambda)^{-t_i} = 0 \Leftrightarrow \sum_{i=1}^{n} x_i (1+\lambda)^{-t_i} = P.$$

Remark 6

If the compounding is continuous, the equation should be

$$\sum_{i=1}^{n} x_i e^{-\lambda t_i} = P.$$

Equation has a solution since the above IRR sufficient conditions holds.

Yield of the bond

- Let us assume that the price the coupon bond is P, the face value is denoted by F, the coupon payments are given $m \geq 2$ times per year (every $1/m$ years).
- Assume that K is the coupon amount per period, and that there are n periods remaining ($n \leq m$).
- The bond yield λ is given by the equation

$$P = F(1 + \lambda/m)^{-n} + \sum_{i=1}^{n} K(1 + \lambda/m)^{-i}.$$

Remark 7

If the compounding is continuous, then the equation to deduce the bond yield λ is

$$P = Fe^{-\lambda n/m} + \sum_{i=1}^{n} Ke^{-\lambda i/m}.$$

Duration

For a bond with price P, and yield λ and cash flow c_i at time t_i for $i = 1, \cdots, n$. The duration of the bond is defined by

$$D := -\frac{1}{P}\frac{\partial P}{\partial \lambda}.$$

- Assume that the bond yield λ is computed with continuous compounding formula, it follows by direct computation that

$$D = \frac{1}{P}\sum_{i=1}^{n} t_i c_i e^{-\lambda t_i}.$$

Remark 8

Duration provides the sensitivity of the bond price with respect to small changes in the yield.

Convexity

The convexity C of the bond is defined by

$$C := \frac{1}{P}\frac{\partial^2 P}{\partial \lambda^2}.$$

In case of the continuous compound bond yield λ, one has by direct computation that

$$C = \frac{1}{P}\sum_{i=1}^{n} t_i^2 c_i e^{-\lambda t_i}.$$

Swaps Contract

- Swap contract is an agreement between two parties to exchange cash flow in the future according to a predetermined formula. There are two basic types of swaps: interest rate and currency.

- An interest rate swap occurs when two parties exchange interest payments periodically. A currency swap is an agreement to deliver one currency against another.

- **Example**: (Currency Swaps) A U.S. company has a British $\pounds$ obligation consisting of $1\pounds$ per year for the next 10 years. The company wants to swap this for a fixed US $\$$ obligation of amount $\$x$. We can refer to $\$x$ as the swap (exchange) rate, because by entering into a swap the company fixes its exchange rate between $1\pounds$ and US $\$$. (At the end of each year, the company will pay to a swap dealer $\$x$ and receive $1\pounds$, so that it can cover its obligation.)

Swaps Contract

- How to compute the swap rate $\$x$?
- We assume the basic data: $r_{\mathsf{US}} = 5.6\%$ and $r_{\mathsf{UK}} = 6.2\%$. The spot exchange rate is $E_0 = \$1.51/\pounds$.
- Equation by computing the PV (present value)

$$x\left[e^{-r_{\mathsf{US}}} + \cdots + e^{-10\times r_{\mathsf{US}}}\right] = E_0\left[e^{-r_{\mathsf{UK}}} + \cdots + e^{-10\times r_{\mathsf{UK}}}\right].$$

Therefore, we get that $x = \$1.47$.