Tutorial 8
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BACK

Questions of today

1. (ch.3 of textbook)Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the extended complex plane. The topology of $\hat{\mathbb{C}}$ is defined as the one point compactification of $\mathbb{C}$, or as the topology of S^2 using stereographic projection(textbook p.88-89). In particular, $\hat{\mathbb{C}}$ is compact. Let U be an open subset of $\hat{\mathbb{C}}$ containing ∞. We say a function $f : U \to \mathbb{C}$ is holomorphic at ∞ if the function g defined by
$$g(z) = \begin{cases} f(\infty), z = 0 \\ f(1/z), z \neq 0 \end{cases}$$
is holomorphic at 0. On the other hand, let $z \in U$ and $f : U \to \hat{\mathbb{C}}$ be a function with $f(z) = \infty$, then we say f is holomorphic at z if and only if the function $1/f$ is holomorphic at z. Show that
   a. An entire function f extended to a holomorphic function on $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ if and only if f is a polynomial.
   b. Show that any holomorphic map from $\hat{\mathbb{C}}$ to itself is a rational function. (Unless f is the constant function with value ∞.)
   c. Show that any biholomorphism from $\hat{\mathbb{C}}$ to itself is a fractional linear transformation.
   d. Show that any non constant holomorphic map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has a zero. Deduce from this the fundamental theorem of algebra.

2. Explain why each of the following open subsets of $\mathbb{C}$ are not conformally equivalent to the open unit disc $\mathbb{D}$.
   a. $\mathbb{D} \cup \{2\}$
   b. $\mathbb{D} \setminus \{0\}$
   c. $\mathbb{C}$

3. There is a map from the set of $\mathrm{SL}(2, \mathbb{C})$ to the set of all fractional linear transformation. Given by
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (f_A : z \mapsto \frac{az+b}{cz+d}).$$
Show that this map is a group homomorphism, with kernel $= \{\pm I\}$.

4. a. Find a conformal mapping from the portion of the unit disc in the first quadrat : $\{x + iy \in \mathbb{D} : x, y > 0\}$ to the upper half plane $\mathbb{H}$.
   b. Find a conformal mapping from $\mathbb{C} \setminus [0, 1]$ to $\mathbb{D} \setminus \{0\}$.

5. a. Show that a conformal mapping from the punctured plane $\mathbb{C} \setminus \{0\}$ to itself must be of the form
$$z \mapsto az^{\pm 1}$$
   b. Show that the punctured plane $\mathbb{C} \setminus \{0\}$ and the punctured disc $\mathbb{D} \setminus \{0\}$ are not conformally equivalent.
   c. Show that $\mathbb{C} \setminus \{0, 1, 2\}$ and $\mathbb{C} \setminus \{0, 1, 3\}$ are not conformally equivalent.

Hints & solutions of today

1. a. Consider the function $g(z) = 1/f(1/z)$, f has no essential singularity at ∞ if and only if g has no essential singularity at 0.
   b. Consider f as a meromorphic function on $\hat{\mathbb{C}}$. Using the compactness of $\hat{\mathbb{C}}$ to show that f has finitely many poles.
   c. Using b. and count the preimage of 0.
   d. f is open by open mapping theorem, and f is closed since the domain is compact.
2. a. Not connected
   b. Not simply connected / the function $1/z$ has nonzero integration over a small circle centered at 0
   c. There is no bounded nonconstant holomorphic functions on $\mathbb{C}$.

3. Skip

4. a. Map the set into $\{z \in \mathbb{H} : 0 < \arg(z) < \pi/2\}$ first.
   b. Apply the transform $z \mapsto 1/z$ first.

5. a. Either f or $1/f$ should have a removable singularity at 0, otherwise f has an essential singularity at 0, and so f is not injective by Carsoti Weierstrass.
   Replcaing f by $1/f$ if necessary, we can assume f is extendable to $\mathbb{C}$. By the same argument, the function $g(z)$ can not have essential singularity at 0, it can neither be a removable singularity, because f is not bounded. As a result, f has a pole at the infinity. The argument in 1a shows that f is a polynomial.
   Finally, f' non zero implies f is of degree ≤ 1. i,e. $f(z) = az + b$. f is nonconstant implies $a \neq 0$. $f(z) \neq 0$ for $z \neq 0$ implies $b = 0$.
   b. f should have a removable singularity at 0.
   c. Show that f extends to a biholomorphic map from $\hat{\mathbb{C}}$ to itself. Hence f must be a fractional linear transform.
