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Office Hour: Send me an email first, then we will arrange a meeting (if you need it).

Solution will be uploaded after the tutorial on Wednesday.

Exercise 1

Show that any finite set in $C(\overline{G})$ is bounded and equicontinuous, where $G \subset \mathbb{R}^n$ is open and bounded.

Solution:

Since continuous function on $\overline{G}$ is uniform continuous (why?), then we consider a finite set $F := \{f_1, ..., f_m\}$ and the goal is to show that it is bounded and equicontinuous.

Since each $f_i \in F$ is continuous, it is uniformly continuous. So, for any $\varepsilon > 0$, there exists some δ_i such that $|f_i(x) - f_i(y)| < \varepsilon$ for all $x, y \in \overline{G}$ and $|x - y| < \delta_i$. Let $\delta := \min\{\delta_1, ..., \delta_m\}$, then

$$|f_i(x) - f_i(y)| < \varepsilon$$

for all $x, y \in \overline{G}$, $|x - y| < \delta$ for all i. So, F is equicontinuous.

Furthermore, since f_i is continuous, $\overline{G}$ is compact, then each f_i is bounded by $\|f_i\|_\infty$ for each i.

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Lemma: All continuous function defined on compact sets $K \subset \mathbb{R}^n$ are uniform continuous.

Proof: Suppose the contrary that $f : K \to \mathbb{R}$ is continuous, but not uniform continuous. Then there exists $\varepsilon_0 > 0$ and two sequences $\{x_n\}, \{y_n\} \subset K$ such that $|x_n - y_n| \to 0$ while $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

Since K is compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x$ as $j \to \infty$ and that $x \in K$. Moreover, consider

$$\lim_{j\to\infty} y_{n_j} = \lim_{j\to\infty} (y_{n_j} - x_{n_j}) + x_{n_j} = 0 + x = x.$$

So both $\{x_n\}$ and $\{y_n\}$ has a convergent subsequence in K.

By continuity of f, we have $f(x_{n_j}) \to f(x)$ and $f(y_{n_j}) \to f(x)$, hence

$$\lim_{j\to\infty} |f(x_{n_j}) - f(y_{n_j})| = 0$$

contradicting the fact that $|f(x_n) - f(y_n)| \geq \varepsilon_0 > 0$.

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Exercise 2

Let $\{f_n\}$ be a sequence of bounded functions in $[0,1]$ and let F_n be

$$F_n(x) = \int_0^x f_n(t)\,dt$$

(a) Show that the sequence $\{F_n\}$ has a convergent subsequence provided there is some M such that $\|f_n\|_\infty \le M$ for all n.

(b) Show that the conclusion in (a) holds when boundedness is replaced by the weaker condition: There is some K such that

$$\int_0^1 |f_n|^2 \le K, \quad \forall n$$

Solution:

(a) F_n is bounded:

$$|F_n(x)| \le \int_0^x |f_n(t)|\,dt \le Mx \le M$$

for all $x \in [0,1]$.

F_n is equicontinuous:

$$|F_n(x) - F_n(y)| \le \int_y^x |f_n(t)|\,dt \le M|x-y|$$

So, $\{F_n\}$ is uniformly bounded and equicontinuous. Then Arzelà-Ascoli's theorem implies that $\{F_n\}$ has a convergent subsequence.

(b) Consider

$$|F_n(x)| \le \int_0^x |f(t)|\,dt \le \left(\int_0^x 1^2\,dt\right)^{\frac{1}{2}} \left(\int_0^x |f_n(t)|^2\,dt\right)^{\frac{1}{2}} \le \sqrt{Kx} \le \sqrt{K}$$

for all $x \in [0,1]$. Moreover

$$\begin{aligned} |F_n(x) - F_n(y)| &\le \int_y^x |f_n(t)|\,dt \\ &\le \left(\int_y^x 1^2\,dt\right)^{\frac{1}{2}} \left(\int_y^x |f_n(t)|^2\,dt\right)^{\frac{1}{2}} \\ &\le \sqrt{K|x-y|} \end{aligned}$$

Then $\{F_n\}$ is equicontinuous.

So, $\{F_n\}$ is uniformly bounded and equicontinuous. The same conclusion applies.

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Exercise 3

Let $K \in C([a,b] \times [a,b])$ and define the operator T by

$$(Tf)(x) = \int_a^b K(x,y)f(y)\,dy$$

(a) Show that T maps $C[a,b]$ to itself.

(b) Show that whenever $\{f_n\}$ is bounded sequence in $C[a,b]$

Solution:

(a) Since $K \in C([a,b] \times [a,b])$, in particular, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|K(x,y) - K(x,' y)| < \varepsilon$ whenever $|x - x'| < \delta$.

Then for $x, x' \in [a,b]$, $|x - x'| < \delta$, we have

$$\begin{aligned} |(Tf)(x) - (Tf)(x')| &\leq \int_a^b |K(x,y) - K(x',y)||f(y)|\,dy \\ &\leq \varepsilon|b-a|\,\|f\|_\infty = M|b-a| \end{aligned}$$

So, $Tf \in C[a,b]$.

(b) We want to make use of the Arzelà-Ascoli's Theorem.

Suppose $\sup_n \|f_n\|_\infty \leq M < \infty$. The the δ in (a) can be chosen so that it is independent of n, say, take minimum among all δ_n's. Then $\{f_n\}$ is equicontinuous.

Furthermore, since

$$|(Tf_n)(x)| \leq \int_a^b |K(x,y)||f(y)|\,dy \leq M|b-a|\,\|K\|_\infty\,,$$

so, $\{f_n\}$ is uniformly bounded.

Then the Arzelà-Ascoli's theorem implies that $\{f_n\}$ has a convergent subsequence.

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Exercise 4

Assuming the knowledge from MATH2230.

Denote the set of holomorphic functions on an open set $U \subset \mathbb{C}$ by $\mathcal{O}(U)$. Then a subset $\mathcal{F} \subset \mathcal{O}(U)$ is said to be *uniformly bounded on compact subsets* of U if for each $K \subset U$ compact, there exists a positive number $B(K)$ such that

$$|f(z)| \leq B(K), \quad \forall f \in \mathcal{F}, z \in K$$

A subset $\mathcal{F} \subset \mathcal{O}(U)$ is said to be *equicontinuous* on a compact set K, if for all $\varepsilon > 0$, there exists δ such that if $z, z' \in K$ and $|z - z'| < \delta$ then

$$|f(z) - f(z')| < \varepsilon, \quad \forall f \in \mathcal{F}$$

A subset $\mathcal{F} \subset \mathcal{O}(U)$ is said to be a *normal family* of holomorphic functions if for any sequence $\{f_n\} \subset \mathcal{F}$, there exists a subsequence $\{f_{n_j}\} \subset \{f_n\}$ such that it converges uniformly on each compact subset of U (Precompact).

Montel's Theorem:
Let U be an open and connected set in $\mathbb{C}$, and let $\mathcal{F}$ be a family of holomorphic functions on U. Suppose that $\mathcal{F}$ is uniformly bounded. Then $\mathcal{F}$ is a normal family on U.

Proof:
We want to apply the Arzelà-Ascoli's theorem, then the result follow immediately. But we will need to show that $\{f_n\}$ is equicontinuous.

Let $K \subset U$ be a compact set. Let $3r$ be the distance from K to U^c. Let $z, z' \in K$ and let C be the circle centered at z' of radius $2r$. Suppose that $|z - z'| < r$, we apply the Cauchy's integral formula

$$\begin{aligned} f(z) - f(z') &= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)} - \frac{f(\zeta)}{\zeta - z'} \, d\zeta \\ &= \frac{z - z'}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)(\zeta - z')} \, d\zeta \end{aligned}$$

then

$$\begin{aligned} |f(z) - f(z')| &= \left| \frac{z - z'}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)(\zeta - z')} \, d\zeta \right| \\ &< \frac{|z - z'|}{2\pi} \frac{2\pi(2r)}{(r)(2r)} \|f\|_\infty \\ &= \frac{\|f\|_\infty}{r} |z - z'| \end{aligned}$$

where $\|f\|_\infty$ is taken over the compact set $K(2r)$, which is the set of all $z \in U$ such that $d(z, K) \leq 2r$. This shows the equicontinuity of $\mathcal{F}$ over K.

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