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Remark: Please let me know if there are typos or mistakes.

Q1

Let (X, d) be a metric space.

(a) If A is a closed subset of X and $x_0 \in X \setminus A$. Show that there is a continuous function f on X such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \in A$.

(b) If A, B are disjoint closed subsets of X. Show that there exists a continuous function g of X such that $g(x) = 1$ for all $x \in A$ and $g(x) = 0$ for all $x \in B$

(c) Given A, B as in part (b), show that there exists disjoint open sets G_1 and G_2 such that $A \subset G_1$ and $B \subset G_2$.

Solution:

(a) Recall the distance of a point $x \in X$ with a subset $A \subset X$ is defined to be

$$d(x, A) := \inf\{d(x, y) : y \in A\}$$

Define $f : X \to \mathbb{R}$ by

$$f(x) := \frac{d(x, A)}{d(x_0, A)}$$

we need to show that f is well-defined, i.e., we want to show that $d(x_0, A) \neq 0$.

Suppose that $d(x_0, A) = 0$, then there exists a sequence $\{x_n\}$ in A such that $d(x_0, x_n) < \frac{1}{n}$. As $n \to \infty$, we have $d(x_0, x_n) \to 0$, meaning that $x_n \to x_0$ w.r.t d. However, since A is closed, we must have $x_0 \in A$. Contradiction. Thus $d(x_0, A) \neq 0$, equivalently, $d(x_0, A) > 0$.

Note that f is continuous which is shown in lecture 7 already. Thus, we see

$$f(x_0) = \frac{d(x_0, A)}{d(x_0, A)} = 1$$

and

$$f(x) = \frac{d(x, A)}{d(x_0, A)} = 0$$

for all $x \in A$ because $d(x, x) = 0$ for all $x \in A$ and hence the infimum must be zero.

(b) Define $g : X \to \mathbb{R}$ by

$$g(x) := \frac{d(x, B)}{d(x, A) + d(x, B)}$$

We now check that g is well-defined, i.e., to check $d(x, A) + d(x, B) > 0$.

Suppose $x \in A$, then $d(x, A) = 0$, while $A \cap B = \emptyset$, hence $d(x, B) > 0$. Similarly for $x \in B$. For $x \notin A$ and $x \notin B$, we have $d(X, A) > 0$ and $d(x, B) > 0$, then $d(x, A) + d(x, B) > 0$. Thus g is well-defined.

Next, we check the continuity of g

We take a sequence of points $\{x_n\}$ from X, for which $x_n \to x$ as $n \to \infty$. By continuity of $d(x, A)$ and $d(x, B)$ as proven in lecture 7, we have the property that $d(x_n, A) \to d(x, A)$ and $d(x_n, B) \to d(x, B)$ as $n \to \infty$. Thus,

$$g(x_n) = \frac{d(x_n, A)}{d(x_n, A) + d(x_n), B)} \to \frac{d(x, A)}{d(x, A) + d(x, B)} = g(x) \text{ as } n \to \infty$$

hence g is continuous.

Lastly, we check the required conditions:

Now that we see, for all $x \in A$,

$$g(x) = \frac{d(x, B)}{d(x, A) + d(x, B)} = \frac{d(x, B)}{0 + d(x, B)} = 1$$

and for all $x \in B$,

$$g(x) = \frac{d(x, B)}{d(x, A) + d(x, B)} = \frac{0}{d(x, A) + 0} = 0$$

(c) Fix $0 < \delta < \frac{1}{2}$, define $G_1 := g^{-1}(1 - \delta, 1 + \delta)$ and $G_2 := g^{-1}(-\delta, \delta)$. They are open since they are preimages of open intervals in $\mathbb{R}$ via g which is continuous.

For all $x \in A$, $g(x) = 1 \in (1 - \delta, 1 + \delta)$, implying that $x \in g^{-1}(1 - \delta, 1 + \delta) = G_1$. Thus $A \subset G_1$.
Similarly, for all $x \in B$, we have $x \in G_2$. Thus $B \subset G_2$.

Now that we show $G_1 \cap G_2 = \emptyset$. Suppose it is not. Then there exists $x \in G_1 \cap G_2$, equivalently, $g(x) \in (1 - \delta, 1 + \delta) \cap (-\delta, \delta)$. By our choice of δ, such an intersection must be empty. Thus contradiction.

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Q2

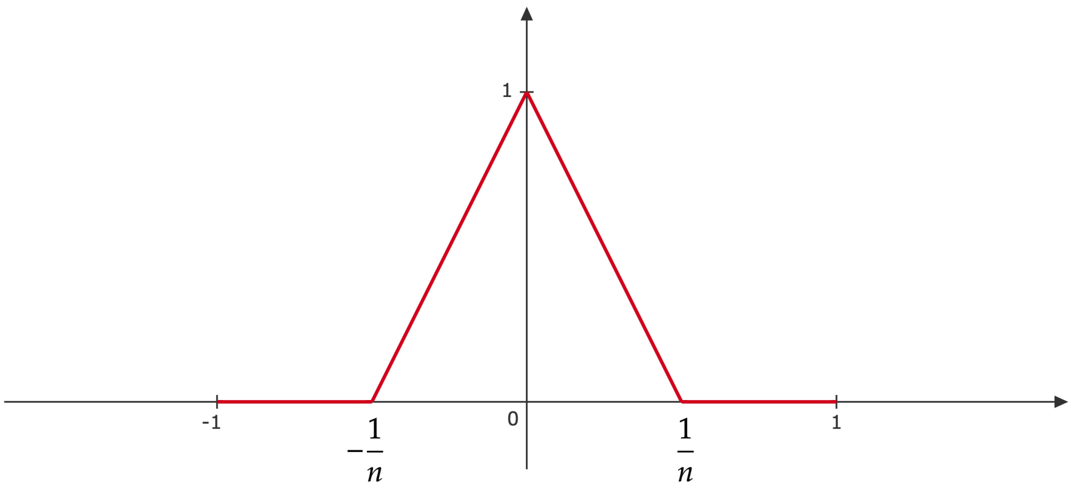
Show that $\Psi : (C[-1,1], d) \to \mathbb{R}$ given by $\Psi(f) = f(0)$ is not a continuous mapping between metric spaces. ($\mathbb{R}$ is always assumed to be equipped with the standard metric $d(x,y) = |x-y|$ for all $x, y \in \mathbb{R}$).

Solution:
The idea is to define a convergent sequence of functions on $(C[-1,1], d)$ such that Ψ does not preserve the convergence.

Define

$$f_n(x) = \begin{cases} -nx+1 & \text{if } x \in [0, \frac{1}{n}] \\ nx+1 & \text{if } x \in [-\frac{1}{n}, 0] \\ 0 & \text{if } x \in [-1, -\frac{1}{n}] \cup [\frac{1}{n}, 0] \end{cases}$$

Graphically,



We can see that $d(f_n, 0) \to 0$ as $n \to \infty$, since

$$d(f_n, 0) = \int_{-1}^{1} |f_n| \, dx = \frac{1}{n}$$

which tends to 0 as $n \to \infty$.

However,

$$\Psi(f_n) = f_n(0) = 1 \neq 0 \text{ as } n \to \infty$$

hence Ψ is not continuous.

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Q3

Recall that $l_2 = \{x = (x_1, x_2, ...) : \sum_{i=1}^{\infty} |x_i|^2 < \infty,\ x_i \in \mathbb{R}\}$. Show that the set

$$H := \left\{ x = (x_1, x_2, ...) : |x_i| \leq \frac{1}{i}, \forall i = 1, 2, ... \right\}$$

is a closed subset in (l_2, d_2).

Solution:
Note that $H \subset l_2$, since for all $x \in H$, $\sum_{i=1}^{\infty} |x_i|^2 \leq \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$.

Take a convergent sequence[1] $\{x^n\}$ from H. Our goal is to show that the limit of $\{x^n\}$, denoted by x, is in H as well.

Since $x_n \to x$ w.r.t, d_2 (defined in HW3), then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have $d_2(x_n, x) < \varepsilon$, equivalently,

$$\left(\sum_{i=1}^{\infty} |x_i^n - x_i|^2 \right)^{\frac{1}{2}} < \varepsilon$$

so, for sufficiently larger n, we have $|x_i^n - x_i| < \varepsilon$ for all i. Therefore

$$||x_i^n| - |x_i|| \leq |x_i^n - x_i| < \varepsilon$$

meaning that

$$|x_i| < |x_i^n| + \varepsilon \leq \frac{1}{i} + \varepsilon$$

since ε is arbitrary, as $\varepsilon \to 0^+$, we have

$$|x_i| \leq \frac{1}{i}$$

hence $x \in H$.

Therefore H is closed.

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[1] Raising the index n as a superscript because x_i is used to denoted the components of $x \in l_2$. If you wish to do research in differential geometry, you need to get used to this style of writing the summation index, because Einstein summation is always used, i.e., instead of writing $\sum_i c_i x_i$, people write $c_i x^i$ to mean summation over i.

Q4

Prove the generalized Hölder's inequality: for all $f_i \in R[a,b]$, $i = 1, ..., n$

$$\int_a^b |f_1 f_2 \cdots f_n| \, dx \le \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_n\|_{p_n}$$

where $p_i > 1$, for all $i = 1, ..., n$, and satisfies $\sum_{i=1}^n \frac{1}{p_i} = 1$.

Solution:

We prove the desired result by induction on n. For the base case, take $n = 2$. then it is the Hölder's inequality, so it holds.

Assume that the inequality holds for $n = k$, i.e.,

$$\int_a^b |f_1 f_2 \cdots f_k| \, dx \le \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_k\|_{p_k}$$

where $p_i > 1$, for all $i = 1, ..., k$ and $\sum_{i=1}^k \frac{1}{p_i} = 1$.

Now consider the $n = k+1$ case. Suppose that $\sum_{i=1}^{k+1} \frac{1}{p_i} = 1$, then define $q = \frac{p_1}{p_1 - 1}$, which is chosen from $\frac{1}{p_1} = 1 - \sum_{i=2}^{k+1} \frac{1}{p_i}$, then $\frac{1}{q} = \sum_{i=2}^{k+1} \frac{1}{p_i} = 1 - \frac{1}{p_1}$.

Thus, by the Hölder's inequality, we have

$$\begin{aligned}
\int_a^b \prod_{i=1}^{k+1} |f_i| \, dx &= \int_a^b |f_1| \prod_{i=2}^{k+1} |f_i| \, dx \\
&\le \|f_1\|_{p_1} \left[\int_a^b \left(\prod_{i=2}^{k+1} |f_i| \right)^q dx \right]^{\frac{1}{q}} \\
&= \|f_1\|_{p_1} \left[\int_a^b \prod_{i=2}^{k+1} |f_i|^q \, dx \right]^{\frac{1}{q}}
\end{aligned}$$

then note that $\sum_{i=2}^{k+1} \frac{q}{p_i} = \sum_{i=2}^{k+1} \frac{p_1}{p_i(p_1-1)} = \frac{p_1}{p_1-1} \sum_{i=2}^{k+1} \frac{1}{p_i} = \frac{p_1}{p_1-1} \left(1 - \frac{1}{p_1}\right) = 1$. So, by the inductive hypothesis $n = k$, we have

$$\begin{aligned}
\int_a^b \prod_{i=1}^{k+1} |f_i| \, dx &\le \|f_1\|_{p_1} \left[\prod_{i=2}^{k+1} \left(\int_a^b |f_i|^{q \frac{p_i}{q}} \right)^{\frac{q}{p_i}} \right]^{\frac{1}{q}} \\
&= \|f\|_{p_1} \prod_{i=2}^{k+1} \left(\int_a^b |f_i|^{p_i} \, dx \right)^{\frac{1}{p_i}} \\
&= \prod_{i=1}^{k+1} \|f_i\|_{p_i}
\end{aligned}$$

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Q5

Show that if $p_2 > p_1 \geq 1$, then there exists a constant $C > 0$ such that

$$\|f\|_{p_1} \leq C \|f\|_{p_2}$$

for all $f \in R[a,b]$.

Solution:
Since $p_2 > p_1$, we have $\frac{p_2}{p_1} > 1$. Let q be the conjugate of $\frac{p_2}{p_1}$, that is, $\frac{1}{q} + \frac{p_1}{p_2} = 1$.

Then we apply the Hölder's inequality

$$\begin{aligned}
\|f\|_{p_1} &= \left(\int_a^b |f|^{p_1} \, dx \right)^{\frac{1}{p_1}} \\
&\leq \left[\left(\int_a^b |f|^{p_1 \frac{p_2}{p_1}} \right)^{\frac{p_1}{p_2}} \left(\int_a^b |1|^q \right)^{\frac{1}{q}} \right]^{\frac{1}{p_1}} \\
&= C \left(\int_a^b |f|^{p_2} \right)^{\frac{1}{p_2}} \\
&= C \|f\|_{p_2}
\end{aligned}$$

where

$$C = \left(\int_a^b |1|^q \right)^{\frac{1}{p_1 q}} > 0$$

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