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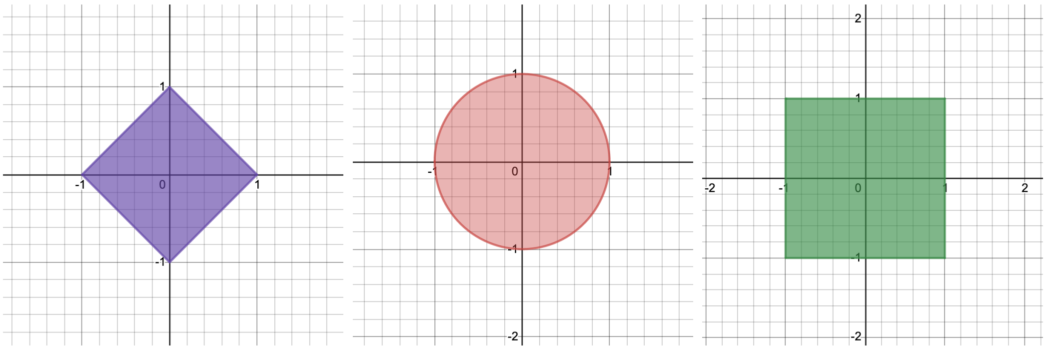
Remark: Please let me know if there are typos or mistakes.

Q1

Sketch the metric ball of radius 1 centered at 0 in $\mathbb{R}^2$ for the metric d_1, d_2 and d_∞ on $\mathbb{R}^2$.

Solution:
Denote $B^p(x, r)$ be the d_p metric ball centered at x with radius r. Denote $x = (x_1, x_2) \in \mathbb{R}^2$.

- For $B^1(0,1) := \{x : d_1(x, 0) \leq 1\} = \{x : |x_1| + |x_2| \leq 1\}$
- For $B^\infty(0,1) := \{x : d_\infty(x, 0) \leq 1\} = \{x : \max\{|x_1|, |x_2|\} \leq 1\}$
- For $B^2(0,1) := \{x : d_2(x, 0) \leq 1\} = \{x : \sqrt{x_1^2 + x_2^2} \leq 1\}$



From left to right: $B^1(0,1)$, $B^2(0,1)$, and $B^\infty(0,1)$

Source: Mathematical Mathematics Memes on Facebook by Markus Klyver.

Q2

Show that for any $\alpha \in \mathbb{R}$, the set

$$\{f \in C[a,b] : f(x) \geq \alpha, \forall x \in [a,b]\}$$

is closed in $(C[a,b], d_\infty)$.

Solution:

Denote $A := \{f \in C[a,b] : f(x) \geq \alpha, \forall x \in [a,b]\}$. To show that A is closed, we show that its complement $B := C[a,b] \backslash A$ is open. Explicitly, $B = \{f \in C[a,b] : f(x) < \alpha, \text{ for some } x \in [a,b]\}$. For any $f \in B$, we want to show that there exists a ball around f such that the ball is contained inside B.

Take any $f \in B$, there exists $x_0 \in [a,b]$ such that $f(x_0) < \alpha$. By continuity of f, there exists a point $y \in [a,b]$ such that $f(y) \leq f(x)$ for all $x \in [a,b]$. Then, we have the relation

$$f(y) \leq f(x_0) < \alpha$$

Let $\varepsilon = \alpha - f(x_0) > 0$. Consider $B^\infty(f,\varepsilon) := \{g \in C[a,b] : d_\infty(g,f) < \varepsilon\}$. We want to show that for all $g \in B^\infty(f,\varepsilon)$, we have $g \in B$. Now, take any $g \in B^\infty(f,\varepsilon)$, we have

$$|g(x_0) - f(x_0)| \leq \max_{x\in[a,b]} |g(x) - f(x)| < \varepsilon = \alpha - f(x_0)$$

.

- If $g(x_0) - f(x_0) \geq 0$, then $g(x_0) - f(x_0) < \alpha - f(x_0) \implies g(x_0) < \alpha$.
- If $g(x_0) - f(x_0) \leq 0$, then $g(x_0) \leq f(x_0) < \alpha \implies g(x_0) < \alpha$.

This shows $g \in B$. Thus B is open, equivalently, $C[a,b] \setminus B = A$ is closed.

■

Remark: A set that is not closed does NOT mean it is open. Some of you wanted to show Q2 by assuming A is open to get a contradiction. This is not true in a general topological space[1]. One example is the *discrete metric space*, in which all sets are both open and closed. Moreover, sets like $[a,b)$ in $\mathbb{R}$ are not open and not closed. Do not confuse with the useful fact that complement of open sets are closed, this does not imply not closed = open. You will definitely see more strange topological spaces when you take MATH3070.

[1] metric spaces are topological spaces

Q3

(a) Let $l_1 = \{x = (x_1, x_2, ...) : \sum_{i=1}^{\infty} |x_i| < \infty, x_i \in \mathbb{R}\}$. Show that $d_1(x, y) := \sum_{i=1}^{\infty} |x_i - y_i|$ is a metric on l_1.

(b) Let $l_2 = \left\{x = (x_1, x_2, ...) : \sum_{i=1}^{\infty} |x_i|^2 < \infty, x_i \in \mathbb{R}\right\}$. Show that $d_2(x, y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^2\right)^{\frac{1}{2}}$ is a metric on l_2.

(c) Let $l_\infty = \{x = (x_1, x_2, ...) : \sup_i |x_i| < \infty, x_i \in \mathbb{R}\}$. Show that $d_\infty(x, y) = \sup_i |x_i - y_i|$ is a metric on l_∞.

(d) Show that the sets $l_1 \subset l_2 \subset l_\infty$.

Solution:

Recall the three axioms of metric:

(i) $d(x, y) \geq 0$ for all $x, y \in X$. Moreover, $d(x, y) = 0 \iff x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

We will first check that the metric is well-defined, then check the three axioms.

(a) Since $x, y \in l_1$, we have $\sum_{i=1}^{\infty} |x_i| < \infty$ and $\sum_{i=1}^{\infty} |y_i| < \infty$. Then

$$d_1(x, y) = \sum_{i=1}^{\infty} |x_i - y_i| \leq \sum_{i=1}^{\infty} (|x_i| + |y_i|) < \infty$$

thus it is well-defined. Then we check the three axioms.

(i) Since $|x_i - y_i| \geq 0$ for all i, then $d_1(x, y) \geq 0$. Moreover, if $x_i = y_i$ for all i, we must have $d_1(x, y) = 0$.

(ii) Since $|x_i - y_i| = |y_i - x_i|$, then $d_1(x, y) = d_1(y, x)$.

(iii) For all $x, y, z \in l_1$, we have $|x_i - y_i| = |x_i - z_i + z_i - y_i| \leq |x_i - z_i| + |z_i - y_i|$ since the series converges, we have $d_1(x, y) \leq d_1(x, z) + d_1(z, y)$.

Thus it is d_1 is a metric on l_1,

(b) Since $x, y \in l_2$, we have $\sum_{i=1}^{\infty} |x_i|^2 < \infty$ and $\sum_{i=1}^{\infty} |y_i|^2 < \infty$. Then consider

$$\sum_{i=1}^{\infty} |x_i - y_i|^2 = \sum_{i=1}^{\infty} |x_i^2 - 2x_i y_i + y_i^2| \leq \sum_{i=1}^{\infty} \left(|x_i|^2 + 2|x_i y_i| + |y_i|^2\right)$$

by Cauchy-Schwarz's inequality, we have

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \sqrt{\sum_{i=1}^{\infty} |x_i|^2} \sqrt{\sum_{i=1}^{\infty} |y_i|^2}$$

then

$$\sum_{i=1}^{\infty} \left(|x_i|^2 + 2|x_i y_i| + |y_i|^2\right) \leq \sum_{i=1}^{\infty} |x_i|^2 + 2\sqrt{\sum_{i=1}^{\infty} |x_i|^2} \sqrt{\sum_{i=1}^{\infty} |y_i|^2} + \sum_{i=1}^{\infty} |y_i|^2$$

and thus

$$\sum_{i=1}^{\infty}|x_i - y_i|^2 = \left(\sqrt{\sum_{i=1}^{\infty}|x_i|^2} + \sqrt{\sum_{i=1}^{\infty}|y_i|^2}\right)^2$$

thus

$$d_2(x,y) = \sqrt{\sum_{i=1}^{\infty}|x_i|^2} + \sqrt{\sum_{i=1}^{\infty}|y_i|^2} < \infty$$

i.e., it is well-defined. Then we check

(i) Since $|x_i - y_i| \geq 0$ for all i, then $d_2(x,y) \geq 0$. Moreover, $d_2(x,y) = 0$ if and only if $x_i = y_i$ for all i.

(ii) Similarly, $|x_i - y_i| = |y_i - x_i|$ for all i, therefore $d_2(x,y) = d_2(x,y)$.

(iii) In the above proof of well-definedness, we know, by similarity, that

$$\sqrt{\sum_{i=1}^{\infty}|x_i + y_i|^2} \leq \sqrt{\sum_{i=1}^{\infty}|x_i|^2} + \sqrt{\sum_{i=1}^{\infty}|y_i|^2}$$

then

$$\begin{aligned} d_2(x,y) &= \sqrt{\sum_{i=1}^{\infty}|x_i - y_i|^2} \\ &= \sqrt{\sum_{i=1}^{\infty}|x_i - z_i + z_i - y_i|^2} \\ &\leq \sqrt{\sum_{i=1}^{\infty}|x_i - z_i|^2} + \sqrt{\sum_{i=1}^{\infty}|z_i - y_i|^2} \\ &= d_2(x,z) + d_2(z,y) \end{aligned}$$

thus d_2 is a metric on l_2.

(c) Since $x, y \in l_\infty$, we know that $\sup_i |x_i| < \infty$ and $\sup_i |y_i| < \infty$. Then

$$d_\infty(x,y) = \sup_i |x_i - y_i| \leq \sup_i (|x_i| + |y_i|) < \infty$$

hence it is well-defined.

(i) Since $|x_i - y_i| \geq 0$ for all i, $\sup_i |x_i - y_i| \geq 0$. Moreover, $x_i = y_i$ for all i if and only if $d_\infty(x,y) = 0$.

(ii) $|x_i - y_i| = |y_i - x_i|$ for all i, then $d_\infty(x,y) = d_\infty(y,x)$.

(iii) $|x_i - y_i| \leq |x_i - z_i| + |z_i - y_i|$ for all i, then taking the supremum yields $d_\infty(x,y) \leq d_\infty(x,z) + d_\infty(z,y)$.

(d) for all $x \in l_1$, we have $\sum_{i=1}^{\infty}|x_i| < \infty$, then this means $\left(\sum_{i=1}^{\infty}|x_i|\right)^2 < \infty$. Moreover, $\sum_{i=1}^{\infty}|x_i|^2 \leq \left(\sum_{i=1}^{\infty}|x_i|\right)^2 < \infty$. Thus $x \in l_2$. Now that $x_2 \in l_2$, we must have $|x_i|^2 < \infty$ for all i, that is $|x_i| < \infty$ for all i. Thus, $\sup_i |x_i| < \infty$, implies $x \in l_\infty$.

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Q4

Let $C^1[a,b] = \{f \in C[a,b] : f \text{ is continuously differentiable on } [a,b]\}$.
Define, for all $f, g \in C^1[a,b]$

$$d(f,g) := \|f - g\|_\infty + \left\|f' - g'\right\|_\infty$$

Show that d is a metric on $C^1[a,b]$. Furthermore, is $f_k(x) := \frac{\sin kx}{k}$, $k = 1, 2, \ldots$ a convergent sequence in $(C^1[0,1], d)$?

Solution:
Explicitly,

$$d(f,g) := \|f - g\|_\infty + \left\|f' - g'\right\|_\infty = \max_{x\in[a,b]} |f(x) - g(x)| - \max_{x\in[a,b]} |f'(x) - g'(x)|$$

(i) Since $|f(x) - g(x)| \geq 0$ and $|f'(x) - g'(x)| \geq 0$ for all $x \in [a,b]$ we have $d(f,g) \geq 0$. Moreover, $|f(x) - g(x)| = 0$ and $|f'(x) - g'(x)| = 0$ for all $x \in [a,b]$

(ii) Since $|f(x) - g(x)| = |g(x) - f(x)|$ and $|f'(x) - g'(x)| = |g'(x) - f'(x)|$ for all $x \in [a,b]$, then $d(f,g) = d(g,f)$.

(iii) $d(f,g) \leq d(f,h) + d(h,g)$ follows from $|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$ and $|f'(x) - g'(x)| \leq |f'(x) - h'(x)| + |h'(x) - g'(x)|$ for all $x \in [a,b]$ as usual.

Since the above holds for all $x \in [a,b]$, it holds for $\max_{x\in[x,b]} |f(x) - g(x)|$ and $\max_{x\in[a,b]} |f'(x) - g'(x)|$ as well.

Now, we want to show whether f_k converges in $(C^1[0,1], d)$. We first observe that $f_k \to 0$ as $k \to \infty$. Then suppose f_k converges to 0 in $(C^1[0,1], d)$, then for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that when $k \geq N$, we have $d(f_k, 0) < \varepsilon$, that is,

$$\max_{x\in[0,1]} \left|\frac{\sin kx}{k}\right| + \max_{x\in[0,1]} |\cos kx| < \varepsilon$$

but $\max_{x\in[0,1]} |\cos kx| = 1$. If we take $\varepsilon = \frac{1}{2}$, then we get a contradiction. Thus f_k does not converges in $(C^1[0,1], d)$. ■

Q5

Let (X_1, d_1) and (X_2, d_2) be two metric spaces. Define $d : (X_1 \times X_2) \times (X_1 \times X_2) \to \mathbb{R}$ by

$$d(u, v) = d_1(x_1, y_1) + d_2(x_2, y_2)$$

for all $u = (x_1, x_2)$ and $v = (y_1, y_2)$ in $X_1 \times X_2$.

(a) Show that d is a metric on $X_1 \times X_2$.

(b) Show that if G_1 is an open set of (X_1, d_1) and G_2 is an open set of (X_2, d_2), then $G_1 \times G_2$ is an open set of $(X_1 \times X_2, d)$.

Solution:

d is well-defined since it is defined as the sum of two metric.

(a) Check the axioms:

(i) Since $d_1(x_1, y_1) \geq 0$ and $d_2(x_2, y_2) \geq 0$ for all $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$, then we have $d(u, v) \geq 0$ for all $u, v \in X_1 \times X_2$. For $u = v$, we have $x_1 = y_1$ and $x_2 = y_2$, then $d(u, v) = 0$ follows from $d_i(x_i, y_i) = 0$ for $i = 1, 2$.

(ii) Symmetry follows from $d_i(x_i, y_i) = d_i(y_i, x_i)$ for $i = 1, 2$.

(iii) Consider $u, v, w \in X_1 \times X_2$, where $u = (x_1, x_2), v = (y_1, y_2), w = (z_1, z_2)$. We know that $d_i(x_i, y_i) \leq d_i(x_i, z_i) + d(z_i, y_i)$. Thus,

$$\begin{aligned} d(u, v) &= d_1(x_1, y_1) + d_2(x_2, y_2) \\ &\leq d_1(x_1, z_1) + d_1(z_1, y_1) + d_2(x_2, z_2) + d_2(z_2, y_2) \\ &= d_1(x_1, z_1) + d_2(x_2, z_2) + d_1(z_1, y_1) + d_2(z_2, y_2) \\ &= d(u, w) + d(w, v) \end{aligned}$$

Thus d is a metric on $X_1 \times X_2$.

(b) Our goal is to show for all $x = (x_1, x_2) \in G_1 \times G_2$, there exists a $\varepsilon > 0$ such that $B(x, \varepsilon) \subset G_1 \times G_2$.

Since G_1 and G_2 are open subsets of X_1 and X_2 respectively, we have

$$\forall x_1 \in G_1, \exists \varepsilon_1 > 0 \text{ such that } B_1(x_1, \varepsilon_1) \subset G$$

$$\forall x_2 \in G_1, \exists \varepsilon_2 > 0 \text{ such that } B_2(x_2, \varepsilon_2) \subset G$$

Let $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. We want to show that for any $x = (x_1, x_2) \in G_1 \times G_2$, the $\varepsilon > 0$ chosen satisfies $B(x, \varepsilon) \subset G_1 \times G_2$.

Now pick any $y \in B(x, \varepsilon)$, we have $d(y, x) < \varepsilon$, that is

$$d_1(x_1, y_1) + d_2(x_2, y_2) < \varepsilon$$

but this implies $d_1(x_i, y_i) < \varepsilon \leq \varepsilon_i$ for $i = 1, 2$. So, $y_1 \in G_1$ and $y_2 \in G_2$, showing $(y_1, y_2) \in G_1 \times G_2$. Hence, $G_1 \times G_2$ is open.

■