Pf of (4)

Let $\delta > 0$ (with $\delta < \frac{\pi}{2}$).

Then $\forall\, n \in \mathbb{N}$, $\exists\, N \in \mathbb{N}$ s.t.

$$N < \frac{(n+\frac{1}{2})}{\pi}\delta \leq N+1 .$$

Clearly, $N \to +\infty$ as $n \to +\infty$.

Now

$$\int_0^\delta |D_n(z)|\,dz = \int_0^\delta \frac{|\sin (n+\frac{1}{2})z|}{2\pi\,|\sin\frac{z}{2}|}\,dz$$

$$= \int_0^{(n+\frac{1}{2})\delta} \frac{|\sin t|}{2\pi\left|\sin\frac{t}{2n+1}\right|}\,\frac{2\,dt}{2n+1} \qquad \left(t = (n+\tfrac{1}{2})z\right)$$

$$= \frac{1}{\pi}\int_0^{(n+\frac{1}{2})\delta} \frac{|\sin t|}{t}\cdot\frac{\left(\frac{t}{2n+1}\right)}{\left|\sin\frac{t}{2n+1}\right|}\,dt$$

$$\geq \frac{1}{\pi}\int_0^{(n+\frac{1}{2})\delta} \frac{|\sin t|}{t}\,dt \qquad \left(\text{Using } \frac{|\sin x|}{x} < 1 \text{ for } 0 < x\right)$$

$$\geq \frac{1}{\pi}\int_0^{\pi N} \frac{|\sin t|}{t}\,dt$$

$$= \frac{1}{\pi}\sum_{k=1}^{N}\int_{(k-1)\pi}^{k\pi} \frac{|\sin t|}{t}\,dt$$

$$= \frac{1}{\pi}\sum_{k=1}^{N}\int_0^{\pi} \frac{|\sin s|}{s+(k-1)\pi}\,ds \qquad \left(s = t-(k-1)\pi\right)$$

$$\geq \frac{1}{\pi}\sum_{k=1}^{N}\int_0^{\pi}\frac{|\sin s|}{k\pi}\,ds \qquad s+(k-1)\pi = t \leq k\pi$$

$$= \frac{1}{\pi^2}\left(\int_0^{\pi}|\sin s|\,ds\right)\cdot\sum_{k=1}^{N}\frac{1}{k}$$

$$= \frac{2}{\pi^2}\sum_{k=1}^{N}\frac{1}{k}$$

Since $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges, and $N \to \infty$ as $n \to \infty$,

we have $\lim_{n\to\infty}\int_0^{\delta}|D_n(z)|\,dz = +\infty$. #

Step 3 Splitting $(S_n f)(x_0) - f(x_0) = I + II$ into integrals concentrated in $[-\delta, \delta]$ & (essential) outside $[-\delta, \delta]$.

By (1) in Step 2,

$$f(x_0) = \int_{-\pi}^{\pi} D_n(z) f(x_0)\,dz$$

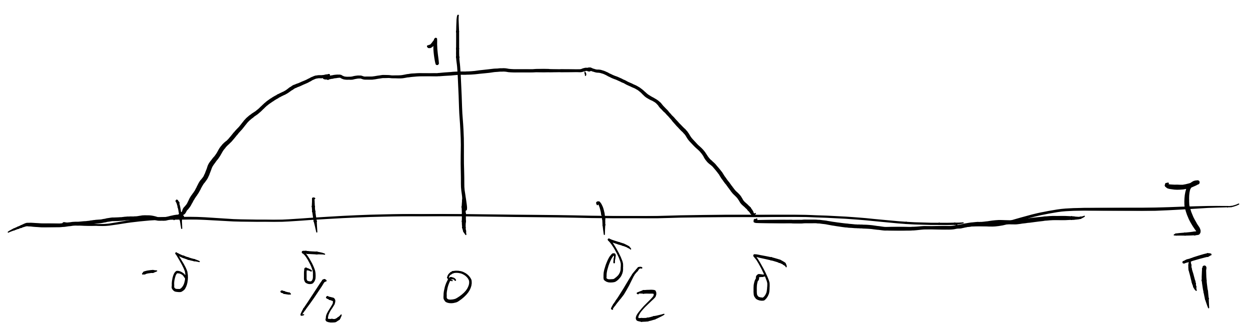
$$\therefore \quad (S_n f)(x_0) - f(x_0) = \int_{-\pi}^{\pi} D_n(z)\left[f(x_0+z) - f(x_0)\right] dz$$

let Φ_δ be a "cut-off" function s.t.

(i) Φ_δ is cts & $0 \le \Phi_\delta \le 1$

(ii) $\Phi_\delta(t) \equiv 1$ for $|t| \le \frac{\delta}{2}$

(iii) $\Phi_\delta(t) \equiv 0$ for $|t| \ge \delta$.



Then $(S_n f)(x_0) - f(x_0)$

$$= \int_{-\pi}^{\pi} D_n(z)\,[f(x_0+z) - f(x_0)]\,dz$$

$$= \int_{-\pi}^{\pi} \Phi_\delta(z) D_n(z)\,[f(x_0+z) - f(x_0)]\,dz$$

$$+ \int_{-\pi}^{\pi} (1 - \Phi_\delta(z)) D_n(z)\,[f(x_0+z) - f(x_0)]\,dz$$

$$= \mathrm{I} + \mathrm{II}$$

where

$$\mathrm{I} = \int_{-\pi}^{\pi} \Phi_\delta(z) D_n(z)\,[f(x_0+z) - f(x_0)]\,dz$$

$$= \int_{-\delta}^{\delta} \Phi_\delta(z) D_n(z)\,[f(x_0+z) - f(x_0)]\,dz$$

and

$$\mathrm{II} = \int_{-\pi}^{\pi} (1-\Phi_\delta(z))\, D_n(z)\, [f(x_0+z)-f(x_0)]\, dz$$

$$= \left(\int_{-\pi}^{-\delta/2} + \int_{\delta/2}^{\pi}\right) (1-\Phi_\delta(z)) D_n(z) [f(x_0+z)-f(x_0)]\, dz$$

<u>Step 4</u>: $\exists\, L>0$ and $\delta_2>0$ such that

$$|\mathrm{I}| \le \frac{4\delta L}{\pi}, \quad \forall\, 0<\delta<\delta_2.$$

<u>Pf</u>: By Lip cts at x_0, $\exists\, L>0$ & $\delta_0>0$ s.t.

$$|f(x_0+z)-f(x_0)| \le L|z|, \quad \forall\, |z|<\delta_0$$

Since $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$, $\exists\, \delta_1>0$ s.t.

$$\left|\frac{\sin\frac{z}{2}}{\frac{z}{2}}\right| > \frac{1}{2}, \quad \forall\, |z|<\delta_1$$

Therefore, for $\delta_2 = \min\{\delta_0, \delta_1\} > 0$,

$$\frac{|f(x_0+z)-f(x_0)|}{|\sin\frac{z}{2}|} \le \frac{L|z|}{\frac{1}{2}\left|\frac{z}{2}\right|} = 4L, \quad \forall\, |z|<\delta_2$$

Hence $\forall\ 0<\delta<\delta_2$, we have

$$|I| \leq \int_{-\delta}^{\delta} |\Phi_\delta(z)|\,|D_n(z)|\,|f(x_0+z)-f(x_0)|\,dz$$

$$= \int_{-\delta}^{\delta} |\Phi_\delta(z)| \frac{|\sin(n+\frac{1}{2})z|}{2\pi\,|\sin\frac{z}{2}|} |f(x_0+z)-f(x_0)|\,dz$$

$$\leq \int_{-\delta}^{\delta} 1\cdot\frac{1}{2\pi}\cdot 4L\,dz = \frac{4\delta L}{\pi}$$ #

<u>Step 5</u>: $\forall\ \varepsilon>0$, $\exists\ \delta>0$ & $n_0>0$ s.t.

$$\frac{4\delta L}{\pi}<\varepsilon/2 \quad \text{and} \quad |II|<\varepsilon/2,\ \forall\ n\geq n_0$$

<u>Pf</u>: $\forall\ \varepsilon>0$, we take

$$\delta=\min\left\{\frac{\varepsilon\pi}{8L}, \delta_2\right\}>0 \quad \left(\begin{array}{c} L\ \&\ \delta_2\ \text{as} \\ \text{in step 4}\end{array}\right)$$

Then $\frac{4\delta L}{\pi}<\varepsilon/2$,

and for this fixed $\delta>0$,

$$II=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1-\Phi_\delta(z)\right)\frac{\sin(n+\frac{1}{2})z}{\sin\frac{z}{2}}\left[f(x_0+z)-f(x_0)\right]dz$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-\Phi_\delta(z))\,[f(x_0+z)-f(x_0)]}{\sin\frac{z}{2}} \left[\sin nz \cos\frac{z}{2} + \cos nz \sin\frac{z}{2}\right] dz$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \left[\frac{(1-\Phi_\delta(z))\,[f(x_0+z)-f(x_0)]}{2\sin\frac{z}{2}} \cos\frac{z}{2}\right] \sin nz\, dz$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} \left[\frac{(1-\Phi_\delta(z))}{2} [f(x_0+z)-f(x_0)]\right] \cos nz\, dz$$

$$= b_n(F_1) + a_n(F_2)$$

where $$F_1(z) = \frac{[1-\Phi_\delta(z)]\,[f(x_0+z)-f(x_0)]\cos\frac{z}{2}}{2\sin\frac{z}{2}}$$

$$F_2(z) = \frac{(1-\Phi_\delta(z))}{2}[f(x_0+z)-f(x_0)].$$

$F_2(z)$ is clearly integrable on $[-\pi,\pi]$

For $F_1(z)$, note that $1-\Phi_\delta(z) = 0$ on $[-\frac{\delta}{2},\frac{\delta}{2}]$ &

$$\left|\sin\frac{z}{2}\right| \geq \sin\frac{\delta}{4} > 0 \quad \text{for } \frac{\delta}{2} \leq |z| \leq \pi.$$

$\Rightarrow$ $F_1(z)$ is also integrable on $[-\pi,\pi]$.

Therefore Riemann-Lebesgue lemma implies

$$\left.\begin{matrix} b_n(F_1) \\ a_n(F_2) \end{matrix}\right\} \longrightarrow 0 \quad \text{as } n \to \infty$$

$\therefore \exists n_0 > 0$ s.t. $|b_n(F_1)| < \frac{\varepsilon}{4}$, $|a_n(F_2)| < \frac{\varepsilon}{4}$ for $n \geq n_0$

$$\therefore \quad |\mathrm{II}| \leq |b_n(F_1)| + |a_n(F_2)| < \frac{\varepsilon}{2}. \quad \divideontimes$$

<u>Final Step</u>: By Steps 3, 4, & 5, we have

$\forall \varepsilon > 0, \exists n_0 > 0$ s.t.

$$\begin{aligned} |S_n f(x_0) - f(x_0)| &= |\mathrm{I} + \mathrm{II}| \\ &\leq |\mathrm{I}| + |\mathrm{II}| \leq \frac{4\delta L}{\pi} + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n > n_0 \end{aligned}$$

$\therefore \quad S_n f(x_0) \to f(x_0)$ as $n \to \infty$ $\divideontimes$

§1.4 Weierstrass Approximation Theorem

(Application of Thm 1.7)

Recall: A cts function defined on $[a,b]$ is piecewise linear if $\exists$ a partition $a = a_0 < a_1 < \cdots < a_n = b$ s.t. f is linear on each subinterval $[a_j, a_{j+1}]$.

Prop 1.11 Let f be a cts function on $[a,b]$. Then $\forall \varepsilon > 0$, $\exists$ a cts, piecewise linear g with $g(a) = f(a)$, $g(b) = f(b)$ such that

$$\|f - g\|_\infty < \varepsilon$$

$$\left(\|f-g\|_\infty = \sup_{[a,b]} |f(x) - g(x)| \right)$$

Pf: f cts on closed interval $[a,b]$

$\Rightarrow$ f uniformly cts $[a,b]$

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \frac{\varepsilon}{2}, \quad \forall\ |x-y| < \delta \quad (x, y \in [a,b])$$

Partition $[a,b]$ into subintervals $I_j = [a_j, a_{j+1}]$

s.t. $\quad |I_j| = a_{j+1} - a_j < \delta, \quad \forall j.$

Define

$$g(x) = \frac{f(a_{j+1}) - f(a_j)}{a_{j+1} - a_j}(x - a_j) + f(a_j), \quad \forall x \in I_j$$

Clearly $g(a_j) = f(a_j)$, $\forall j$. In particular $g(a) = f(a)$ & $g(b) = f(b)$, and $g(x)$ is piecewise linear on $[a,b]$.

(and cts)

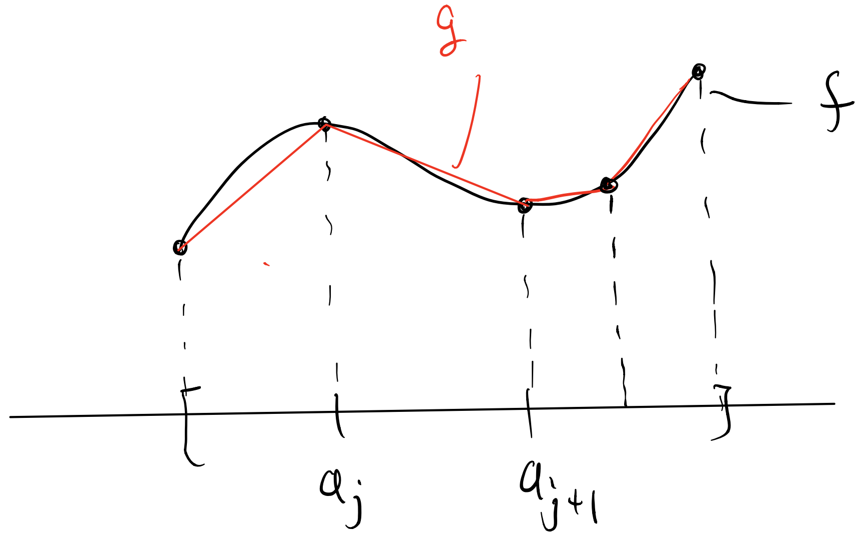
Then $\forall x \in I_j \subset [a,b]$

$$|f(x) - g(x)| = \left| f(x) - \frac{f(a_{j+1}) - f(a_j)}{a_{j+1} - a_j}(x - a_j) - f(a_j) \right|$$

$$\leq |f(x) - f(a_j)| + \frac{|f(a_{j+1}) - f(a_j)|}{a_{j+1} - a_j}(x - a_j)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore \quad \sup_{\cup I_j} |f(x) - g(x)| < \varepsilon$ ie $\|f - g\|_\infty < \varepsilon$. #



Terminology: A trigonometric polynomial is of the form $P(\cos x, \sin x)$, where $P(x,y)$ is a polynomail of 2 variables.

Note that a trigonometric polynomial is a finite Fourier series and vice-versa. (Ex!)

[finite Fourier series: $a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$, $N < \infty$]

Prop 1.12 Let f be cts function on $[0, \pi]$. Then $\forall \varepsilon > 0$, $\exists$ a trigonometric polynomial h s.t. $\|f - h\|_\infty < \varepsilon$

Pf: Extend f to $[-\pi, \pi]$ by

$$f(x) = \begin{cases} f(x), & x \in [0, \pi] \\ f(-x), & x \in [-\pi, 0] \end{cases} \quad \text{(even extension)}$$

Then this extension is cts on $[-\pi, \pi]$ & $f(\pi) = f(-\pi)$, hence extends to a 2π-periodically cts function on $\mathbb{R}$.

By Prop 1.11, $\forall \varepsilon > 0$, $\exists$ piecewise linear (cts) g on $[-\pi, \pi]$ s.t. $\|f - g\|_\infty < \frac{\varepsilon}{2}$ &

$$g(\pi) = f(\pi) = f(-\pi) = g(-\pi)$$

$\Rightarrow$ g extends to a piecewise linear 2π-periodic function $\tilde{g}$ on $\mathbb{R}$.

Clearly $\tilde{g}$ satisfies a Lip condition. (why?)

Then Thm 1.7 $\Rightarrow$ $\exists N > 0$ s.t.

$$\|g - S_N g\|_\infty < \frac{\varepsilon}{2} \quad \left(\begin{array}{c} S_N g \to g \\ \text{uniformly} \end{array} \right)$$

Therefore $\|f - S_N g\|_\infty \le \|f - g\|_\infty + \|g - S_N g\|_\infty$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

$\therefore$ $h = S_N g$ is the required trigonometric polynomial $\#$

Thm 1.13 (**Weierstrass Approximate Theorem**)

Let $f \in C[a,b]$ Then $\forall \varepsilon > 0$, $\exists$ a polynomial g s.t.

$$\|f - g\|_\infty < \varepsilon.$$

Pf: Consider $[a,b] = [0,\pi]$ first.

Extend f to $[-\pi,\pi]$ as in prop 1.12

$\forall \varepsilon > 0$, choose trigonometric polynomial

$h = p(\cos x, \sin x)$ s.t.

$$\|f - h\|_\infty < \frac{\varepsilon}{2}$$

Using the fact that

$$\begin{cases} \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ \sin x = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} \end{cases}$$

converge uniformly

$\exists N > 0$ s.t.

$$\left\| h(x) - p\left(\sum_{n=1}^{N} \frac{(-1)^n x^{2n}}{(2n)!}, \sum_{n=1}^{N} \frac{(-1)^n x^{2n-1}}{(2n-1)!} \right) \right\|_\infty < \frac{\varepsilon}{2}$$

Clearly $q(x) = p\left(\sum_{n=1}^{N} \frac{(-1)^n x^{2n}}{(2n)!}, \sum_{n=1}^{N} \frac{(-1)^n x^{2n-1}}{(2n-1)!} \right)$

is the required polynomial s.t. $\|f-g\|_\infty < \varepsilon$.

For general $[a,b]$, $\varphi(x) = f\left(\frac{b-a}{\pi} x + a\right) \in C[0,\pi]$

$\Rightarrow$ $\exists$ $q(x)$ polynomial s.t.

$$\|\varphi(x) - q(x)\|_\infty < \varepsilon \text{ on } [-\pi, \pi]$$

$\Rightarrow$ $q\left(\frac{\pi}{b-a}(x-a)\right)$ is the polynomial s.t.

$$\left\| f(x) - q\left(\frac{\pi}{b-a}(x-a)\right) \right\|_\infty < \varepsilon$$