MATH 2230A - HW 9 - Students' Samples: Mistake and Good Work, Q3b

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3. In each case, find the order m of the pole and the corresponding residue B at the singularity $z = 0$:

$(a)\ \dfrac{\sinh z}{z^4}; \qquad (b)\ \dfrac{1}{z(e^z - 1)}.$

Ans. $(a)\ m = 3,\ B = \dfrac{1}{6}; \quad (b)\ m = 2,\ B = -\dfrac{1}{2}.$

1. I am really curious: many (nearly half) of you have given the following non-standard (and un-necessarily complicated) answer using the geometric series:

(b) $f(z) = \dfrac{1}{z(e^z-1)} = \dfrac{\phi(z)}{z} \Rightarrow \phi(z) = \dfrac{1}{e^z-1}$

Singularity : $z=0$.

Note that $\phi(z)$ is not defined at $z=0$, so we should use Laurent series.

Maclaurin series $e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!} = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots$

$$\frac{1}{z(e^z-1)} = \frac{1}{z\left(z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)} = \frac{1}{z^2\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\cdots\right)}$$

Take $|z| \le 1$. Let $R(z) = \dfrac{z}{2!} + \dfrac{z^2}{3!} + \cdots$

Then $\left|\dfrac{z}{2!} + \dfrac{z^2}{3!} + \cdots\right| = \left|\sum_{n=2}^{\infty} \dfrac{z^{n-1}}{n!}\right| \le \sum_{n=2}^{\infty} \dfrac{|z|^{n-1}}{n!} \le \sum_{n=2}^{\infty} \dfrac{1}{n!} = \sum_{n=0}^{\infty} \dfrac{1}{n!} - 2 = e - 2 < 1.$

Therefore, we can write $\dfrac{1}{1+\left(\frac{z}{2!}+\frac{z^2}{3!}+\cdots\right)} = \dfrac{1}{1+R(z)} = 1 - R(z) + R(z)^2 + \cdots$

$$\text{Then, } \frac{1}{z(e^z-1)} = \frac{1}{z^2}\left(1 - R(z) + R(z)^2 + \cdots\right)$$
$$= \frac{1}{z^2} - \frac{R(z)}{z^2} + \frac{R(z)^2}{z^2} - \frac{R(z)^3}{z^2} + \cdots$$
$$= \frac{1}{z^2} - \left(\frac{1}{2!}z^{-1} + \frac{1}{3!} + \cdots\right) + \cdots$$

Rest of the terms do not have negative powers of z.

Hence, the isolated singular point $z=0$ is a pole of order $m=2$.

$B = \operatorname*{Res}_{z=0}(f(z)) = -\dfrac{1}{2}$.

The proof is in valid until the very last line: marks will be deducted if you did not state explicitly where the terms in the last equality comes from. I expect you to state clearly that the principal parts come *only* from the first term $\frac{R(z)}{z^2}$ and does NOT come from the higher power terms of $R(z)$, that is, $\frac{(R(z))^n}{z^2}$ where $n \geq 2$. Otherwise, I would regard that the principal part comes from considering **a combination of terms** from $(R(z))^n$, but this in general could not been done until we have proved that the collection of terms in each $(R(z))^n$ converges unconditionally. Or to be precise, you have to show that the (geometric) sum of a sequence of unconditionally convergent (Taylor) series is again unconditionally convergent. I do think that is true in this case but I expect related verifications.

That says, I could only (merely) accept if you write something like

$\forall z \in D(0;1)$

$$|h(z)| \leq \sum_{n=1}^{+\infty} \frac{1}{(n+1)!} = e-2 < 1.$$

Hence $\forall z \in D(0;1)\setminus\{0\}$.

$$\frac{1}{z(e^z-1)} = \frac{1}{z^2}\cdot\frac{1}{1+h(z)} \qquad (|h(z)|<1)$$

$$= \frac{1}{z^2}\left[1+(-h(z))+[-h(z)]^2+\cdots\right]$$

$$= \frac{1}{z^2} - \left(\frac{1}{2z}+\frac{1}{3!}+\frac{z}{4!}\right) + \frac{1}{z^2}\left\{[-h(z)]^2+\cdots\right\}$$

This also applies to $\mathbb{C}\setminus\{0\}$ by series Theory.

Since $\frac{1}{z^2}\left\{[-h(z)]^2+[-h(z)]^3+\cdots\right\}$ contains terms with positive power of z only

The principal part is

$$\frac{1}{z^2}-\frac{1}{2z}$$

$\therefore z_0 = 0$ is a pole of order 2 of $\frac{1}{z(e^z-1)}$

with residue $-\frac{1}{2}$.

The student here stated clearly that the principal parts only come from the first term. In fact to be precise, it is NOT enough to justify that the higher order terms does not contain negative powers just by looking at its terms as it is now an a summation of terms in which every term is again an infinite sum. Things may get weird. A safe way to see why those higher power terms does not contain the principal parts is to prove that it is holomorphic at 0. This can be shown by writing the higher power terms as $\frac{(g(z))^2}{z^2(1+g(z))}$ by its convergence which have been shown by students. Then one could write something like

$$\frac{1}{z(e^z-1)} = \frac{1}{z^2} - \frac{1}{2z} + \text{ some functions holomorphic at } 0$$

to conclude the result.

2. In fact, there are many more simpler methods to compute residues. The following student considered the derivatives of related power series:

b) $$f(z) = \frac{1}{z(e^z-1)} = \frac{\frac{1}{e^z-1}}{z} = \frac{\frac{1}{\sum_{n=1}^{\infty} \frac{z^n}{n!}}}{z}$$

$$= \frac{\frac{1}{z^1+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots}}{z}$$

$$= \frac{\frac{1}{1+\frac{z}{2!}+\frac{z^2}{3!}+\cdots}}{z^2}$$

$\therefore$ Since $\frac{1}{1+\frac{z}{2!}+\frac{z^2}{3!}+\cdots}$ is analytic and non-zero at $z=0$, So $f(z) = \frac{\phi(z)}{z^2}$

where $\phi(z) = \left(1+\frac{z}{2!}+\ldots\right)^{-1}$

$\therefore$ $z=0$ is a pole with order 2

$$\phi'(z) = -\left(1+\frac{z}{2!}+\ldots\right)^{-2}\left(\frac{1}{2}+\frac{2}{3!}z+\ldots\right)$$

$$\phi'(0) = -\frac{1}{2}$$

$\therefore$ $B = \frac{\phi'(0)}{1!} = -\frac{1}{2}$ //

3. A probably even simpler method is to use long division as if the power series are polynomials. This method is standard.

(b) Note that $e^z - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!} \Rightarrow \frac{1}{z(e^z-1)} = \frac{1}{z(z+\frac{z^2}{2}+\frac{z^3}{6}+\cdots)} = \frac{1}{z^2(1+\frac{z}{2}+\frac{z^2}{6}+\cdots)}$

$$1+\frac{z}{2}+\frac{z^2}{6}+\cdots \,\overline{)\,1\,}\quad = 1-\frac{z}{2}-\frac{z^2}{12}+\cdots$$

$$\begin{array}{l} 1 \\ \underline{1+\frac{z}{2}+\frac{z^2}{6}+\cdots} \\ -\frac{z}{2}-\frac{z^2}{6}+\cdots \\ \underline{-\frac{z}{2}-\frac{z^2}{4}+\cdots} \\ \frac{z^2}{12}+\cdots \\ \underline{\frac{z^2}{12}+\cdots} \\ +(\text{some term above } z^3) \end{array}$$

$= \left(\frac{1}{z^2}\right)\left(1-\frac{z}{2}-\frac{z^2}{12}+(\text{some term above } z^3)\right)$

$= z^{-2} - \frac{z^{-1}}{2} + \frac{1}{12} + \text{some term above } z$

Note $\frac{1}{z(e^z-1)}$ is analytic in the punctured disk $D(0;r_0)\setminus\{0\}$ when $r_0>0$, $\forall n<2$, $a_{-n}=0$, $a_{-2}\neq 0$

Hence, we have $z=0$ is the pole of order 2 and residue is $\frac{-1}{2}$.

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4. This student even provides some theoretical background for the long division. In fact it follows from the fact the we could compute product of Taylor series by considering its Cauchy Product, that is, as if they are polynomials. (Please see Sec. 73 in the textbook for details.)

The function $f(z) = e^z - 1$ is entire. $f(z) = 0 \Leftrightarrow e^z = 1 \Leftrightarrow e^x \cos y = 1$ and $e^x \sin y = 0$
$\Leftrightarrow x = 0$ and $y = 2n\pi \Leftrightarrow z = 2n\pi i$ for $n \in \mathbb{Z}$. Then, 0 is an isolated singularity.
The Taylor series expansion of f at 0 is $f(z) = \left(\sum_{n=0}^{\infty} \frac{1}{n!} z^n\right) - 1 = \sum_{n=1}^{\infty} \frac{1}{n!} z^n = z \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$
which is analytic on $\mathbb{C}$.
Now, $\frac{1}{e^z - 1} = \frac{1}{z} \cdot \frac{1}{g(z)}$ for $z \neq 2n\pi i$, where $g(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots$.
Since g is analytic at 0 and $g(0) = 1 \neq 0$, $\exists \delta > 0$ such that $\frac{1}{g(z)}$ is analytic on $D(0, \delta)$.
Let $\frac{1}{g(z)} = \sum_{n=0}^{\infty} a_n z^n$ as its Taylor series at $z_0 = 0$.
Now, $1 = g(z) \sum_{n=0}^{\infty} a_n z^n = \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots\right)(a_0 + a_1 z + a_2 z^2 + \dots)$.
By comparing coefficients, we have $a_0 = 1$, $\frac{a_0}{2!} + a_1 = 0$, $\frac{a_0}{3!} + \frac{a_1}{2!} + a_2 = 0$, etc., ...
then $a_0 = 1$, $a_1 = -\frac{1}{2}$, $a_2 = \frac{1}{12}$.
For $z \neq 0$, we have $\frac{1}{z(e^z - 1)} = \frac{1}{z^2} \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^{n-2} = z^{-2} - \frac{1}{2} z^{-1} + \frac{1}{12} + \sum_{n=3}^{\infty} a_n z^{n-2}$,
the principal part is $z^{-2} - \frac{1}{2} z^{-1}$. Then 0 is a pole of order 2, with residue $-\frac{1}{2}$.

5. Lastly, I attach here my solution to the question using the Residue formula (please refer to HW9 solution or the Lecture note for details): this is in fact similar to the long division
Let $f(z) = \frac{1}{z(e^z - 1)}$. Let $g(z) = e^z - 1$. Note that $g(0) = 0$ but $g'(0) = 1 \neq 0$. Hence, 0 is a zero of order 1 of g. It is clear that 0 is a order-1 zero of $z \mapsto z$. Hence $z(e^z - 1)$ has a zero of order 2 at 0 which implies f has a pole of order 2 at 0. By the Residue Formula that follows readily by considering Laurentz Series (Proposition 0.4 in this Solution, or Formula 1.117 in Lecture Note), we have

$$\mathrm{Res}(f, 0) = \left.\frac{d}{dz}\right|_{z=0} \frac{h(z)}{1!} = h'(0)$$

where $h(z) = z^2 f(z) = \frac{z}{e^z - 1}$ locally at 0 and is holomorphic non-zero at 0. Note $h(z)(e^z - 1) = z$. By differentiating both sides, we have $h'(z)(e^z - 1) + h(z)e^z = 1$, which implies $h(0) = 1$. Differentiating once more, we have $h''(z)(e^z - 1) + h'(z)e^z + h'(z)e^z + h(z)e^z = 0$, which imples $2h'(0) + h(0) = 0$ and $h'(0) = -1/2$. Therefore, $\mathrm{Res}(f, 0) = -1/2$.