The Chinese University of Hong Kong
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Tutorial 2: Elementary Functions of Complex Numbers
(Prepared by Lam Ka Lok)

1 Subsets on the Complex Plane

A function consists of a domain and a rule. We shall recall basic subsets of complex numbers that are to be domains of functions.

Definition 1.1. Let $S \in \mathbb{C}$. The we have the following definitions:

1. We call S open if for all $z \in U$, there exists $r > 0$ such that the open ball $B(z, r) := \{w \in \mathbb{C} : |w - z| < r\}$ centered at z with radius r lies in S

2. We call S closed if its complement is open.

3. The smallest closed set containing S is called its closure and is denoted by $\overline{S}$ while the largest open set contained in S is called its interior and is denoted by S^o.

4. We call S bounded if $S \subset B(0, r)$ for some $r > 0$. Equivalently, there exists $M > 0$ such that $|z| \leq M$ for all $z \in S$

5. We call S compact if S is closed and bounded.

6. We call S is connected, or path-connected, if for all $z, w \in S$, there exists a continuous curve (function) $\gamma : [0, 1] \to S$ such that $F(0) = z$ and $F(1) = w$, that is, connecting z, w.

7. We call S simply-connected, if S is path-connected and any two continuous curves can be continuously deform to another (or intuitively S is path-connected and has "no holes").

Remark. The first and the last few definitions (open-ness and connectedness)are of the most important in this course. Please refer to Math3070 for any ambiguity of the above definitions.

Example 1.2. *Below are some basic examples of subsets that are with certain properties.*

1. *Let $z \in \mathbb{C}, r > 0$. Then the open balls $B(z, r)$ are bounded, open sets.*

2. *Let $z \in \mathbb{C}, r > 0$. Define $\overline{B(z, r)} := \{w \in \mathbb{C} : |w - z| \leq r\}$ to be the closed ball centered at z with radius r. Then these closed balls are closed sets and are in fact the closure of open balls $B(z, r)$. Since they are bounded as well, they are indeed compact sets.*

3. *Let $z \in \mathbb{C}, r_1, r_2 > 0$. Define the open annulus $A(z, r_1, r_2) := \{w \in \mathbb{C} : r_1 < |w - z| < r_2\}$. Then $A(z, r_1, r_2)$ is open, path-connected, but is not simply connected.*

4. *We call a subset $K \subset \mathbb{C}$ convex if for all $x, y \in K$, we have $tx + (1 - t)y \in \mathbb{C}$ for all $t \in [0, 1]$. Every convex set is path-connected and in fact also simply connected.*

2 Elementary Functions

Unless otherwise specfied, we shall denote the domain of a function an open subset $U \subset \mathbb{C}$.

Definition 2.1. Let $f : U \to \mathbb{C}$ be a function. Then we call the functions $\mathrm{Re}(f) : U \to \mathbb{R}$ and $\mathrm{Im}(f) : U \to \mathbb{R}$ defined by $\mathrm{Re}(f)(z) = \mathrm{Re}(f(z))$ and $\mathrm{Im}(f)(z) = \mathrm{Im}(f(z))$ the real and imaginary part of f respectively.

Definition 2.2 (Polynomial Functions)**.** Let $n \in \mathbb{N}$. Let $a_0, \dots, a_n \in \mathbb{C}$ with $a_n \neq 0$. We call the function $P_n : U \to \mathbb{C}$ a polynomial function of degree n if it is defined by

$$P_n(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$$

We call further $a_0, \dots, a_n$ the coefficients of P_n.

Remark. A polynomial is constructed through a finite step of addition and product to the constant and identity functions.

Definition 2.3 (Rational Functions)**.** Let $P, Q : U \to \mathbb{C}$ be polynomial functions. Suppose Q is non-zero on U. The quotient $\frac{P}{Q}$ is well-defined and we call it a rational function.

Remark. By definition, any polynomial function is a rational function. To define a rational function, one has to ensure that the polynomial on the denominator is never zero on its domain.

Example 2.4. *Let $f : U \to \mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We call $z \in U$ a zero of f if $f(z) = 0$.Let $f : \mathbb{C} \to \mathbb{C}$ be an affine function, that is, a polynomial function of order 1. Then the zeros of* $\mathrm{Im}(f)$ *form a straight line.*

Definition 2.5 (Exponential, Trigonometric and Hyperbolic Functions)**.** Recall that the exponential function $\exp : \mathbb{C} \to \mathbb{C}$ is given by $e^z := e^x e^{iy}$ if $z = x + iy$ for $x, y \in \mathbb{R}$. Note that e^{iy} is further defined as $\cos y + i \sin y$ where $y \in \mathbb{R}$ by the Euler Formula. We can define the trigonometric and hyperbolic functions using the exponential functions for all $z \in F$ as follows:

a) $\cos z := \frac{e^{iz} + e^{-iz}}{2}$

b) $\sin z := \frac{e^{iz} - e^{-iz}}{2i}$

c) $\cosh z := \frac{e^{z} + e^{-z}}{2}$

d) $\sinh z := \frac{e^{z} - e^{-z}}{2}$

Remark. The definitions of trigonometric and hyperbolic functions are directly generalized from the case where the domain is $\mathbb{R}$.

It is extremely important to note that different from the real case, the complex exponential function is *not* injective. We need to do more to construct the "inverse" function, that is, the logarithmic function for complex numbers.

Definition 2.6 (Complex Logarithms)**.** Let $a_0 \in \mathbb{R}$. We call the interval $(a_o, a_0 + 2\pi]$ a branch. We call the function $\log : \mathbb{C} \backslash \{0\}$ defined by $\log z := \ln |z| + i \arg z$, where $\arg z \in (a_o, a_0 + 2\pi]$, the logarithmic function with respective to the branch $(a_o, a_0 + 2\pi]$.

Remark. If the branch is $(-\pi, \pi]$, we call this the principle branch and the respective logarithmic function the principle logarithmic function, denoted by Log. Note that for every branch, the respective logarithmic function is an inverse of the complex exponential function.
Without a pre-defined branch, the notation $\log z$ denotes a set instead.

Definition 2.7 (Power functions)**.** Let $a_0 \in \mathbb{R}$ and $(a_o, a_0 + 2\pi]$ a branch. Let $c \in \mathbb{C}$. We can define $z^c := e^{c \log z}$. The function $z \mapsto z^c$ is called a power function with index c, which is defined on $\mathbb{C} \backslash \{0\}$

Remark. Same as the case of logarithmic functions, if there is no pre-defined branch, then the notation z^c denotes a set in general.

3 Continuous Functions

We do not study arbitrary functions. We study functions that preserve structures.

Definition 3.1. Let $f : U \to \mathbb{C}$ be a function. We say f is continuous at $z_0 \in U$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ if $z \in U$ and $|z - z_0| < \delta$.

Theorem 3.2. *Let $f : U \to \mathbb{C}$ be a function. Let $z_0 \in U$ Then the following are equivalent:*

1. *f is continuous at z_0*
2. *$\operatorname{Re} f$ and $\operatorname{Im} f$ are continuous at z_0*
3. *$f(z_n) \to f(z_0)$ for all sequence $z_n \in U$ such that $z_n \to z_0$*

Definition 3.3. We call $f : U \to \mathbb{C}$ a continuous function if it is continuous for all $z_0 \in U$.

Theorem 3.4. *Denote $C(U)$ the space of continuous functions from U to $\mathbb{C}$. Then $C(U)$ satisfies the following:*

1. *$f + g \in C(U)$ if $f, g \in C(U)$*
2. *$fg \in C(U)$ if $f, g \in C(U)$*
3. *$kf \in C(U)$ if $f \in C(U), k \in \mathbb{C}$*
4. *$\frac{f}{g} \in C(U)$ if $f, g \in C(U)$ and g is nonzero on U.*
5. *If $U = \mathbb{C}$, then $g \circ f \in C(U)$ if $f, g \in C(U)$*

The first three shows that the space of continuous functions is a $\mathbb{C}-$ algebra

Example 3.5. *The following are basic examples of continuous functions:*

1. *Identity and the constant function are continuous on $\mathbb{C}$.*
2. *Polynomial functions are continuous on $\mathbb{C}$ by 1 and the fact that the space of continuous functions on $\mathbb{C}$ is a $\mathbb{C}$- algebra*
3. *Rational functions are continuous on $\mathbb{C}$ except where the denominator is zero by 2 and the fact that $C(U)$ is closed in quotient.*
4. *Exponential function is continuous on $\mathbb{C}$ by considering its real and imaginary part.*
5. *Trigonometric and hyperbolic functions are continuous on $\mathbb{C}$ since they are just linear combination of the exponential functions.*

Example 3.6. *The following is an extremely important counter-example of continuous function. Consider the principle branch $(-\pi, \pi]$. Then the principle logarithmic function $\operatorname{Log} z := \ln|z| + i \operatorname{Arg} z$ is not continuous on the line $\theta = \pi$, that is, the negative real-axis.*

Let's end this note with the following powerful facts concerning continuous functions.

Theorem 3.7. *Let $f : U \to \mathbb{C}$ be a continuous function. Then we have*

1. *$f(U)$ is connected if U is connected.*
2. *$f(U)$ is compact (closed and bounced) if U is compact.*

Corollary 3.8 (Extreme Value Theorem)**.** *Let $f : U \to \mathbb{C}$ be a continuous function from a closed and bounded (compact) domain U. Then we have $\sup f(U) = \max f(U)$ and $\max f(U) < \infty$*

4 Exericse

1. Show that the set of open sets on $\mathbb{C}$ satisfies the following properties:

1. Intersection of finitely many open sets is open.
2. Union of arbitrarily many open sets is open.

2. Recall that a subset $K \subset \mathbb{C}$ is convex if for all $z, w \in K$ and $t \in [0,1]$ we have $tz + (1-t)w \in K$.

1. Show that for all $r > 0$, $z \in \mathbb{C}$. The open ball $B(z,r)$ is convex.
2. Show that all convex sets are path-connected. This shows all open balls are path-connected.

3. Let $a, b \in \mathbb{C}$ with $a \neq 0$. Consider the affine function, that is, the order 1 polynomial function, $f(z) = az + b$. Show that the zeros of f is a singleton while the zeros of $\operatorname{Im}(f)$ is a straight line.

4. Let $P(z), Q(Z) : \mathbb{C} \to \mathbb{C}$ be two affine functions.

(i) If any, give an example such that the zeros of $\operatorname{Im}(\frac{P(z)}{Q(z)})$ is a circle.

(ii) If any, give an example such that the zeros of $\operatorname{Im}(\frac{P(z)}{Q(z)})$ is a straight line.

(iii) If any, give an example such that the zeros of $\operatorname{Im}(\frac{P(z)}{Q(z)})$ is neither a circle nor a straight.

5. Find all the zeros of the complex polynomials.

a) $z^3 + 1$ b) $z^5 - i$ c) $z^n - 1$ where $n \in \mathbb{N}$

6. Consider the principle branch. Compute the value of the following:

a) $\operatorname{Log}(-1 + \sqrt{3}i)$ b) i^i c) $(1+i)^i$

7. Let $a, b, z, w \in \mathbb{C}$. Verify if the following are always true:

(i) $z^a z^b = z^{a+b}$

(ii) $(z^a)^b = z^{ab}$

(iii) $(zw)^a = z^a w^a$

8. We are considering the complex trigonometric functions $\sin z, \cos z : \mathbb{C} \to \mathbb{C}$ in this exercise.

(i) Show that $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$

(ii) Show that $\cos z$ and $\sin z$ are not injective.

(iii) Show that $\cos z$ and $\sin z$) are surjective, that is, the range of them is the whole complex plane. (Hint: Try to solve $\cos z = 1$ first)

(iv) Hence, find an inverse function for both $\sin z$ and $\cos z$.

9. In this exercise, we are considering the continuity of the conjugate operation. Define $f : \mathbb{C} \to \mathbb{C}$ by $z \mapsto \overline{z}$. Show that

(i). f is continuous

(ii). Hence, the functions $z \mapsto \operatorname{Re}(z)$, $z \mapsto \operatorname{Im}(z)$, $z \mapsto |z|^2$ on the whole space.

(iii). Define $g(z) = \frac{z}{\overline{z}}$ on $\mathbb{C}\backslash\{0\}$. Does there exist a continuous extension of g to the whole space? That is, can we define g further on 0 such that this new g is continuous at 0?

10. Let $a_0 \in \mathbb{R}$. Consider the logarithmic function over the branch $(a_0, a_0 + 2\pi]$. Show that it is not continuous on the straight line $\{z \in \mathbb{C} : a_0 \in \arg z\}$. We call this straight line (ray) the *branch cut* of the branch.