MATH 2230A - HW 7 - Solutions

Full solutions at P.170-171 Q2, 4, 10; P.196 Q6

Commonly missed steps in Purple and common mistakes at the back

The Cauchy Integral Formulae (again) and the Taylor's Theorem play a crucial role in this HW. Through the solution, we use $B(x,r), \overline{B(x,r)}, C(x,r)$ to denote open balls, closed balls and circles (boundaries of balls) respectively.

Theorem 0.1 (Cauchy Integral Formula)**.** *Let $f : \Omega \to \mathbb{C}$ be holomorphic on a closed simply connected domain Ω. Let $z \in \Omega^o$, the interior of Ω, then we have*

$$2\pi i f(z) = \int_{\partial\Omega} \frac{f(w)}{w-z} dw$$

Theorem 0.2 (Generalized Cauchy Integral Formula)**.** *Let $f : \Omega \to \mathbb{C}$ be holomorphic on a closed simply connected domain Ω. Then f is infinitely differentiable on Ω. Furthermore we have for all $n \in \mathbb{N}$ and $z \in \Omega^o$, the interior of Ω that*

$$\frac{1}{n!} f^{(n)}(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(w)}{(w-z)^{n+1}} dw$$

Theorem 0.3 (Taylor's Theorem)**.** *Let $z_0 \in \mathbb{C}$ and $r > 0$. Let $f : \overline{B(z_0, r)} \to \mathbb{C}$ be holomorphic on $\overline{B(z_0, r)}$. Then f is a power series on $B(z_0, r)$. In particular, for all $z \in B(z_0, r)$, we have*

$$f(z) = \sum_{i=0}^{\infty} \frac{f^n(z_0)}{n!} (z - z_0)^n$$

which is a power series centered at z_0. Note that f^n exists for all $n \in \mathbb{N}$ on $B(z_0, r)$ by the generalized Cauchy Integral Formula and $f^{(n)}(z_0)$ could be computed by the formula as well.

Remark. In fact it suffices to assume that f is holomorphic on $B(z_0, r)$ instead of its closure $\overline{B(z_0, r)}$. The latter in fact implies the former (which is demonstrated in the solution of P.196 Q6). Nonetheless, to be in line with the assumption of Cauchy Integral Formulae, the former is written here (as in Lecture Note Sec 14.1).

Remark. Every time, we would be giving full solutions to *selected* problems only. Other problems are provided with partial solutions. Please feel free to contact us if you need help on the solutions.

P.170 - 171

1. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(a) $\displaystyle\int_C \frac{e^{-z}\, dz}{z - (\pi i/2)}$; (b) $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}\, dz$; (c) $\displaystyle\int_C \frac{z\, dz}{2z+1}$;

(d) $\displaystyle\int_C \frac{\cosh z}{z^4}\, dz$; (e) $\displaystyle\int_C \frac{\tan(z/2)}{(z-x_0)^2}\, dz \quad (-2 < x_0 < 2)$.

Ans. (a) 2π; (b) $\pi i/4$; (c) $-\pi i/2$; (d) 0; (e) $i\pi \sec^2(x_0/2)$.

Solution. For 1(d). Take $f(z) = \cosh z$ and $w = 0$ to apply the generalized Cauchy Integral Formula. For 1(e). Take $f(z) = \tan(z/2)$ and $w = x_0$ to apply the generalized Cauchy integral formula. Please do not forget to verify why you could use the generalized Cauchy Integral Formula on the chosen $f(z)$ and w.

2. Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

(a) $g(z) = \dfrac{1}{z^2+4}$; (b) $g(z) = \dfrac{1}{(z^2+4)^2}$.

Ans. (a) $\pi/2$; (b) $\pi/16$.

Solution. We write $g(z) = 1/(z^2+4) = 1/(z+2i)^2(z-2i)^2$ by factorizing the denominator. Let $f(z) := 1/(z+2i)^2$ and $w := 2i$. Then by standard arguments considering the denominator, f is holomorphic everywhere except at $-2i$. Note that $|-2i-i| = 3 > 2$, so $-2i \notin \overline{B(i,2)}$. Hence f is holomorphic on $\overline{B(i,2)}$. In addition, $|w-i| = |2i-i| = 1 < 2$. Hence $w \in B(i,2)$. Therefore, the generalized Cauchy Integral Formula can be applied on f and w. We then have,

$$\int_C g(z)dz = \int_C \frac{f(z)}{(z-w)^2}dz = 2\pi i f'(w) = 2\pi i f'(2i) = 2\pi i \left[\frac{-2}{(z+2i)^3}\right]_{z=2i} = \frac{\pi}{16}$$

4. Let C be any simple closed contour, described in the positive sense in the z plane, and write

$$g(z) = \int_C \frac{s^3+2s}{(s-z)^3}\,ds.$$

Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z) = 0$ when z is outside.

Solution. When z is inside C, that is , in the interior of the region bounded by C: let $f(s) := s^3+2s$. Since f is polynomial, f is entire and hence is holomorphic on the closed region bounded by C, which is simply connected. By z is inside C. Hence, we can apply the generalized Cauchy integral formula to have

$$g(z) = \int_C \frac{s^3+2s}{(s-z)^3}ds = \int_C \frac{f(s)}{(s-z)^3}ds = \frac{2pii}{2!}f^{(2)}(z) = \pi i\,[6s]_{s=z} = 6\pi i z.$$

When z is outside C: let $h_z(s) = s^3+2s/(s-z)^3$. Note that h is holomorphic everywhere except at z, but z is outside C. Therefore, h is holomorphic everywhere in the closed region bounded by C. Since C is simply connected, we can apply the Cauchy-Goursat Theorem to have $g(z) = \int_C h_z(s)ds = 0$.

5. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)\,dz}{z-z_0} = \int_C \frac{f(z)\,dz}{(z-z_0)^2}.$$

Solution. The question does not make sense in the context for this course if f' is not analytic on the closed region bounded by C. We assume it is (by perhaps assuming that f is analytic on a simply connected open set containing the contour C). Then both are zero by the Cauchy Goursat Theorem.

6. Let f denote a function that is *continuous* on a simple closed contour C. Following the procedure used in Sec. 56, prove that the function

$$g(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s-z}$$

is *analytic* at each point z interior to C and that

$$g'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2}$$

at such a point.

Solution. Just follow the procedure in Section 56.

10. Let f be an entire function such that $|f(z)| \leq A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.

Suggestion: Use Cauchy's inequality (Sec. 57) to show that the second derivative $f''(z)$ is zero everywhere in the plane. Note that the constant M_R in Cauchy's inequality is less than or equal to $A(|z_0| + R)$.

Solution. **Method 1**

We first show that the second derivative of f is zero everywhere. Since f is entire, by the generalized Cauchy integral formula, f is infinitely complex differentiabel everywhere. In particular, f'' exists everywhere on $\mathbb{C}$. We proceed to show $f''(z) = 0$ for all $z \in \mathbb{C}$.

Fix $z \in \mathbb{C}$. Take $R \in R$ such that $R > |z|$. Since f is entire, f is holomorphic on $\overline{B(0,R)}$. As $z \in B(0,R)$, using the generalized Cauchy Integral formula, the triangle inequality and the assumption, we then have

$$\begin{aligned}
\frac{2\pi i}{2!}\left|f^{(2)}(z)\right| &= \left|\int_{C(0,R)} \frac{f(w)}{w-z}dw\right| \\
&\leq \int_{C(0.R)} \frac{|f(w)|}{|w-z|}|dw| \\
&\leq \int_{C(0,R)} \frac{A|w|}{|w|-|z|}|dw| = \int_{C(0,R)} \frac{AR}{(R-|z|)^3}|dw| = \frac{2\pi AR^2}{(R-|z|)^3}
\end{aligned}$$

where $C(0,R)$ is the circle centered at 0 with radius R. Note that $\frac{2\pi AR^2}{(R-|z|)^3} = \frac{2\pi A1/R}{(1-|z|/R)^3} \to 0$ as $R \to \infty$. Since the above inequality is true for all $R > |z|$, we have by Sandwich theorem that $\left|f^{(2)}(z)\right| = 0$, which implies $f^{(2)}(z) = 0$. Since z is arbitrary, we have $f'' = 0$ everywhere on $\mathbb{C}$.

It remains to show that f is the desired form. Since $f'' = 0$, we have f' is a constant (which could be proved by using the Cauchy-Riemann Equations to extend similar property on real derivatives to the complex case). As anti derivatives differ by a constant (which could also be proved via the CR equations) and linear polynomial clearly has constant derivatives, we can conclude that f is a linear polynomial, that is $f(z) = a_1 z + a_2$ for some $a_1, a_2 \in \mathbb{C}$ for all $z \in \mathbb{C}$.

Lastly, by the assumption, we have $|f(0)| \leq A|0|$. Hence $f(0) = 0$, which imples $a_2 = 0$. We then conclude that $f(z) = a_1 z$ for all $z \in \mathbb{C}$.

Method 2

We apply the Cauchy's inequality directly to show that $f'' = 0$. Let $z \in \mathbb{C}$ and $R > 0$. Since f is an entire function, f is analytic on $\overline{B(z,R)}$; we can apply Cauchy's inequality to have that

$$\left|f^{(2)}(z)\right| \leq \frac{2!}{R^2} M_R$$

where $M_R := \max_{\omega \in C(z,R)} |f(w)|$. Note that by triangle inequality, we have for all $\omega \in C(z,R)$ that $|w| = |w - z + z| \leq R + |z|$. Hence by the assumption, we have

$$\left|f^{(2)}(z)\right| \leq \frac{2!}{R^2} M_R = \frac{2}{R^2} \max_{\omega \in C(z,R)} |f(w)| \leq \frac{2}{R^2} \max_{\omega \in C(z,R)} A|w| \leq \frac{2A}{R^2} \max_{\omega \in C(z,R)} R + |z| = \frac{2A(R+|z|)}{R^2}$$

Note that $\lim_{R\to\infty} \frac{2A(R+|z|)}{R^2} = 0$. Since the inequality holds for all $R > 0$, we have by Squeeze theorem, $\left|f^{(2)}(z)\right| = 0$, which implies $f^{(2)}(z) = 0$. As z is arbitrary, we have $f'' = 0$. The remaining is the same as Method 1 above.

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5. Use the identity $\sinh(z + \pi i) = -\sinh z$, verified in Exercise 7(*a*), Sec. 39, and the fact that $\sinh z$ is periodic with period $2\pi i$ to find the Taylor series for $\sinh z$ about the point $z_0 = \pi i$.

$$Ans. \; -\sum_{n=0}^{\infty} \frac{(z-\pi i)^{2n+1}}{(2n+1)!} \qquad (|z - \pi i| < \infty).$$

Solution. Use the hints to compute the derivatives for $\sinh(z)$ at $z = \pi i$. The result follows from simple computations.

6. What is the largest circle within which the Maclaurin series for the function $\tanh z$ converges to $\tanh z$? Write the first two nonzero terms of that series.

Solution. Let $f(z) = \tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-1}}$. We first compute the non-differentiable points of $f(z)$. Let $z \in \mathbb{C}$. Then by considering the denominator f is not differentiable at z if and only if we have

$$e^z + e^{-z} = 0 \Leftrightarrow e^{2z} = -1 \Leftrightarrow 2z \in \log(-1) \Leftrightarrow 2z = (2n+1)\pi i \exists n \in \mathbb{N}$$

For all $n \in \mathbb{N}$, define $z_n := (2n+1)\pi i/2$. By the above $\{z_n\}$ is precisely the set of non-differentiable points.

We proceed to claim that the Maclaurin series of f converges (point-wise) on $B(0, \pi/2)$. First notice that $|z_n| \geq \pi/2$ for all $n \in \mathbb{N}$. Since $\{z_n\}$ is precisely the set of non-differentiable points, f is holomorphic on $\overline{B(0, r)}$ for all $r < \pi/2$. By Taylor's theorem, the Maclaurin series, that is, the Taylor series centered at 0 converges on $B(0, r)$ for all $r < \pi/2$. Therefore the Maclaurin series converges on $\bigcup_{r<\pi/2} B(0, r) = B(0, \pi/2)$.

Next, we claim that $B(0, \pi/2)$ is the largest circle within which the Maclaurin series converges. Let $\epsilon > 0$. Note that f is not differentiable at $z_0 = \pi i/2$ where $|z_0| = \pi/2 < \pi/2 + \epsilon$. Hence f is not differentiable everywhere within any slight enlargement $B(0, \pi/2 + \epsilon)$. It follows that $B(0, \pi/2)$ is the largest required circle.

Lastly, we compute the first two non-zero terms of the Maclaurin Series. Using $f(z) = \tanh z, f'(z) = \operatorname{sech}^2(z), f^{(2)}(z) = -2\operatorname{sech}^2(z)\tanh(z), f^{(3)}(z) = -2\operatorname{sech}^4(z) + 4\operatorname{sech}^2(z)\tanh^2(z)$, (please note the negative sign in the second derivative) we can compute that $f(0) = 0, f^2(0) = 0$ and the first two non-zero terms being

$$f'(0)z + \frac{f^{(3)}(0)}{3!}z^3 = z - \frac{2z^3}{3!}z^3 = z - \frac{1}{3}z^3$$

Remark. As mentioned in the beginning of this solution, the Taylor's Theorem used here assumes require analyticity on a *closed* disk (as in the Lecture Note Sec 14.1). With that requirement, you CANNOT directly apply the Taylor's theorem on $B(0, \pi/2)$ to conclude the Maclaurin series converges inside the circle. It is because $f(z)$ is NOT holomorphic on $\overline{B(0, \pi/2)}$!
Nonetheless, it should be reminded that Taylor's theorem could require only analyticity on an *open* disk (as in the textbook Sec 62). The two formulations are equivalent. In fact careful readers should see that the transition between the formulations is demonstrated in the above proof.

7. Show that if $f(z) = \sin z$, then

$$f^{(2n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = (-1)^n \qquad (n = 0, 1, 2, \ldots).$$

Thus give an alternative derivation of the Maclaurin series (3) for $\sin z$ in Sec. 64.

Solution. It follows from simple computations.

Common Misktakes:

1. Some of you misunderstood the boundedness condition of a function and so misused the Liouville's Theorem: a function is bounded on a domain Ω if there exists a real constant $M > 0$ such that $|f(z)| \leq M$ for all $z \in \Omega$. The constant M is independent of z. Therefore the condition in P.170 Q10: $\exists A > 0$ such that for all $z \in \Omega$ $|f(z)| \leq A|z|$ does NOT imply that f is bounded. As the "bound" $A|z|$ is dependent on the point z.

2. There is no mention of the Taylor's Theorem in almost all of your homework. After the Fundamental Theorem of Cantour Integrals (concerning antiderivatives), Cauchy-Goursat Theorem and the Cauchy Integral Formulae, the Taylor's Theorem (and the Laurent's Theorem) is the next major theorem to have a solid understanding. You should remember their assumptions and conclusions by heart so that you know when to use and not to use them.

3. Please revise the hyperbolic functions as well as techniques for solving exponential/ trigonometric equations. Many of you have missed a minus side in the third derivatives of $\tanh(z)$.