MATH 2050A - HW 4 - Comments and Common Mistakes:

Common Misktakes:

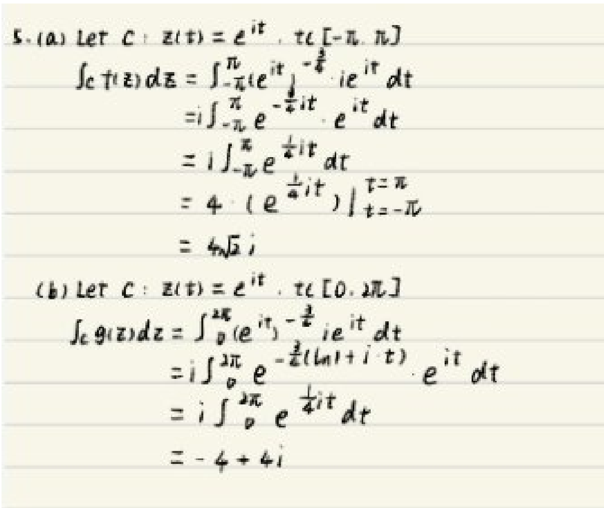
Question 5:

Let C denote the positively oriented unit circle $|z| = 1$ about the origin.

(a). Let $f(z) = z^{-3/4}$ using the principal branch. Show that $\int_C f(z)dz = 4\sqrt{2}i$

(b). Let $g(z) = z^{-3/4}$ using the branch $0 < \arg z \leq 2\pi$. Show that $\int_C g(z)dz = -4 + 4i$

1. Below is a typical answer from all of you (for 5a):



5.(a) Let $C: z(t) = e^{it}$, $t \in [-\pi, \pi]$

$\int_C f(z)dz = \int_{-\pi}^{\pi} (e^{it})^{-\frac{3}{4}} ie^{it} dt$

$= i\int_{-\pi}^{\pi} e^{-\frac{3}{4}it} \cdot e^{it} dt$

$= i\int_{-\pi}^{\pi} e^{\frac{1}{4}it} dt$

$= 4 \cdot (e^{\frac{1}{4}it}) \Big|_{t=-\pi}^{t=\pi}$

$= 4\sqrt{2}i$

(b) Let $C: z(t) = e^{it}$, $t \in [0, 2\pi]$

$\int_C g(z)dz = \int_0^{2\pi} (e^{it})^{-\frac{3}{4}} ie^{it} dt$

$= i\int_0^{2\pi} e^{-\frac{3}{4}(\ln 1 + i \cdot t)} \cdot e^{it} dt$

$= i\int_0^{2\pi} e^{\frac{1}{4}it} dt$

$= -4 + 4i$

The student starts by parametrizing the contour $\|z\| = 1$ and then use the definition of contour integral to turn the contour integral into an ordinary integral on $\mathbb{R}$. The problem is here. The function $f(z) = z^{-3/4}$ is a power function which is not continuous on the branch cut and hence at the endpoints. As a result the integrand above only represents the function behavior everywhere except at the endpoint. To be precies, the above integral is not $f \circ \gamma \cdot \gamma'$ if we denote γ the parametrization of the contour. You can check that we have

$$f(\gamma(-\pi)) = f(-1) = (-1)^{-3/4} = e^{-3i/4arg(-1)} = e^{-3i\pi/4 \cdot \pi} = e^{-3i\pi/4}$$

but $e^{i(-\pi)-3/4} = e^{3\pi i/4} \neq i = f(\gamma(-\pi))$. (Note that $\arg(-1) = \pi$ as the pincipal branch is used.)

2. To overcome the above problem, one can either compute using improper integrals or explicity state that the above integrand is almost the same as $f \circ \gamma \cdot \gamma'$ and so their integral should be the same. Please refer to the solution to see how improper integrals are used.

Question 7:

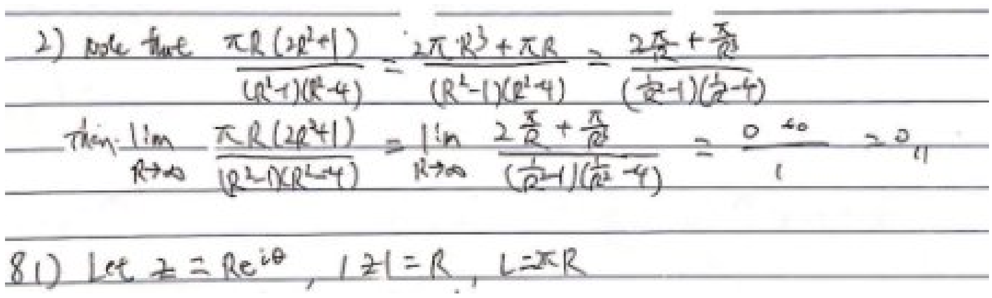
Let $R > 2$. Denote the *upper half* of the circle $|z| = R$, C_R, oriented counter-clockwise.

(i). Show that

$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} dz \right| \le \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}$$

(ii). Show that the value of the integral tends to 0 as $R \to \infty$.
Hint: Divide the numerator and denominator on the right by R^4.

1. The performance in this question is good. Many students knew how to use the Triangle inequalities (both for ordinary complex numbers and integrals) for part (i) and Squeeze Theorem to do part (ii).
2. Some students were very careless on part (ii):



2) Note that $\frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)} = \frac{2\pi R^3+\pi R}{(R^2-1)(R^2-4)} = \frac{2\frac{\pi}{R}+\frac{\pi}{R^3}}{(\frac{1}{R^2}-1)(\frac{1}{R^2}-4)}$

Then $\lim_{R\to\infty} \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)} = \lim_{R\to\infty} \frac{2\frac{\pi}{R}+\frac{\pi}{R^3}}{(\frac{1}{R^2}-1)(\frac{1}{R^2}-4)} = \frac{0}{1} = 0$ //

8i) Let $z = Re^{i\theta}$, $|z| = R$, $L = \pi R$

The question asks you to show the integral tends to 0, but not to show the limit of the RHS tends to 0. You have to establish the relation.

Question 11:

Show that

$$\int_{-1}^{1} z^i dz = \frac{1+e^{-\pi}}{2}(1-i)$$

where z^i uses the principal branch and the path of integration is any contour from -1 to 1 that lies above the real axis (except for its endpoints).

1. This question tests your understanding on the Fundamental Theorem of Contour Integral (Theorem 1.34 in Lecture Note), which concerns about the relation between the existence of antiderivatives and path independence. In other words, the question tests you when you can compute an integral just by computing difference of two numbers. Your performance shows to me that (nearly) all of you did not understand, or I should say, *did not care* about the condition for computing a contour integral via antiderivatives.

2. First, a number of you did not read the question carefully. The question asked you to check the contour integral from EVERY path connecting -1 and 1. If you considered only 1 path without any explanation, you would receive no mark from this question.

Q11 Note $z^i = e^{i \operatorname{Log} z}$
$= e^{-\theta} e^{i \ln r}$
Then parametric representation is $C = \{z = x : -1 \leq x \leq 1\}$.
Hence $\int_{-1}^{1} z^i dz = \int_{-1}^{0} z^i dz + \int_{0}^{1} z^i dz$.
$= \int_{-1}^{0} e^{-\pi} e^{i \ln(-x)} dx + \int_{0}^{1} e^{0} e^{i \ln x} dx$ No.
$= \int_{0}^{1} e^{-\pi} e^{i \ln x} dx + \int_{0}^{1} e^{i \ln x} dx$

3. Second, in fact most of you did check the integral for all paths. However, most of you, again, did the question by considering the anti-derivative of the power function *which uses the principal branch*! That way, most of you have been trapped by the question: you CANNOT apply the Fundamental Theorem of Contour Integral directly on the function because the function is not continuous at the end-point (branch cut) of the path. No anti-derivative for the function exists at the endpoint as well. Please refer to Theorem 1.34 in Lecture Notes for the detailed condition for applying antiderivatives to compute integrals.

4. Third, only a few students tried to overcome the above problem by considering another branch with $-\pi/2 < \arg z \leq 3\pi/2$. I planned to give marks as long as you verified why the Fundamental Theorem of Contour Integral could be used on the new function branch (with the new branch) by writing *at least* that the new function is continuous and has anti-derivative on the contour. (Please refer to the Lecture Note for the FULL Requirement: the contour lies in some open connected set (domain) on which the function is continuous and has an anti-derivative). Nonetheless, *no one* mentioned the existence of anti-derivative on the contour.

5. Lastly, in fact after verifying that the Fundamental Theorem of Contour Integral can be applied on the new function (with the new branch), one still have to check that the integral of the new function is the same as that of the function in question (with the principal branch). For instance, why don't we consider the branch $3\pi/2 < \arg z \leq 7\pi/2$ instead? This is because the choice of the branch $-\pi/2 < \arg z \leq 3\pi/2$ makes sure that the new function and the original function in the question are in fact *the same* function on every contour except maybe on the end-points as they both take arguments from $(0, \pi)$. That way, their contour integrals are the same.

6. In fact, Professor Yu has added an Example in the Lecture Note to demonstrate how to tackle such questions. Please see P. 32, Example 4 after Theorem 1.34 in the Lecture Note for details. You may check my solution as well.

General Comments:

1. The graded problems are Q5, 7 and 11, which carries 3, 3, 4 points respectively.
2. HW4 aims at testing your understanding to contour integrals. Key concepts include
   (a) computing contour integrals using suitable parametrizations with the help of the Fundamental Theorem of Calculus in $\mathbb{R}$
   (b) computing contour integrals on points of discontinuity (branch cuts)
   (c) using the modulus integral inequality (or triangle inequality for contour integral)
   (d) the use of the Fundamental Theorem of Contour Integrals (Theorem 1.34 in Lecture Notes) which concerns the relation between the existence of Anti-derivatives and the computation of contour integrals via path independence

   Please look carefully at the related theorems concerning the above concepts in the lecture notes.
3. Once again, I feel that *most* of you did not pay attention to details of the condition in theorems (and problems). For example, in this HW, whenever computing contour integrals, nearly all of you just find an antiderivative to compute as if every integral can be computed using path independence, ignoring every continuity condition required for you to do that. This is NOT in DSE or in high schools: we DO NOT require you to always find the "correct" answer, but we want you to always make a "well-justified" attempt to your answer.
4. To emphasize the justification of an answer instead of finding the correct answer, my grading focuses on the rigor of your answer instead of the truth: you may get very low marks even though you have reached the answer the question asks you to reach ,while you may full marks even though you make wrong computations as long as you have shown your understanding on Theorems in the lecture notes.
5. From 2xxx onwards, courses in MATH often focus on of proofs instead of answers. Please keep that in mind and good luck to your midterm.