MATH 2230A - HW 3 - Solutions

Full solutions at Q1,5,7

Remark. Every time, we would be giving full solutions to *selected* problems only. Other problems are provided with partial solutions. Please feel free to contact us if you need help on the solutions.

1 (P.61-62 Q8)**.** Using the method in Example 2, Sec. 19, show that $f'(z)$ does not exist at any $z \in \mathbb{C}$ if

a) $f(z) = \mathrm{Re}(z)$

b) $f(z) = \mathrm{Im}(z)$

Solution.

a). Let $z \in \mathbb{C}$. Then $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{\mathrm{Re}(h)}{h}$. Take sequences (x_n), (y_n) in $\mathbb{C}$ with $x_n, y_n \neq 0$, $\mathrm{Re}(x_n) = 0$ and $\mathrm{Im}(y_n) = 0$ for all $n \in \mathbb{N}$ such that $x_n \to 0$ and $y_n \to 0$. (For example, we can simply take $x_n = i\frac{1}{n}$ and $y_n = \frac{1}{n}$ for $n \in \mathbb{N}$.) Then $\frac{\mathrm{Re}(x_n)}{x_n} = 0 \to 0$ and $\frac{\mathrm{Re}(y_n)}{y_n} = 1 \to 1$. By sequential criteria for limit, the limit $\lim_{h\to 0} \frac{\mathrm{Re}(h)}{h}$ does not exist and hence $f'(z)$ does not exists. Since z is arbitrary, f is not differentiable for all $z \in \mathbb{C}$

b). Follow the above argument but we consider $\lim_{h\to 0} \frac{\mathrm{Im}(h)}{h}$ instead with some minor changes.

2 (P.61-62 Q9)**.** Let f be a function on $\mathbb{C}$ defined by

$$f(z) = \begin{cases} \frac{\overline{z}^2}{z} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

Consider f to map from the z-plane to the w- plane. Consider f at the point $z_0 = 0$. Denote $\Delta z := z - z_0$, and hence $\Delta w, \Delta x, \Delta y$ as usual

(i). Show that $\frac{\Delta w}{\Delta z} = 1$ at each nonzero point on the real and imaginary axes in the Δz plane.

(ii). Show that $\frac{\Delta w}{\Delta z} = -1$ on the line $\Delta y = \Delta x$ in the Δz plane.

(iii). Hence, show that $f'(0)$ does not exist.

Solution. You can do the question bascially just by following the steps.

3 (P.70-71 Q1)**.** Using the theorem in Sec. 21, show that $f'(z)$ does not exist at any point $z \in \mathbb{C}$ if f is defined by

a) $f(z) = \overline{z}$

b) $f(z) = z - \overline{z}$

c) $f(z) = 2x + ixy^2$

d) $f(z) = e^x e^{-iy}$

where x, y denote $\mathrm{Re}\, z, \mathrm{Im}\, z$ respectively.

Solution. Consider the Cauchy-Riemann equations in each case and find some points where the C-R equations are not satisfied.

4 (P.70-71, Q2)**.** Using the theorem in Sec. 23, for each of the following functions f on $\mathbb{C}$

(i). show that $f'(z)$ and its derivative $f''(z)$ exist everywhere on $\mathbb{C}$

(ii). find $f''(z)$.

a) $f(z) = iz + 2$

b) $f(z) = e^{-x}e^{-iy}$

c) $f(z) = z^3$

d) $f(z) = \cos x \cosh y - i \sin x \sinh y$

where x, y denote $\operatorname{Re} z, \operatorname{Im} z$ respectively.

Solution.

(i). For the existence of $f'(z)$, consider the partial derivatives of f in each case. Show that they are continuous everywhere and satisfy the C-R equations.
For $f''(z)$, note that $f' = \partial_x(\operatorname{Re} f) + i\partial_x(\operatorname{Im} f)$. Do the routine checking afterwards: partial derivatives of f' are continuous and satisfy the CR equations everywhere.

(ii). The answer follows from the equality $f'' = \partial_x(\operatorname{Re} f') + i\partial_x(\operatorname{Im} f')$

5 (P.70-71, Q3)**.** Using theorems in Sec. 21 and 23, for each of the following functions f on $\mathbb{C}$, determine where $f'(z)$ exists and find the corresponding value for such z.

a) $f(z) = \frac{1}{z}$

b) $f(z) = x^2 + iy^2$

c) $f(z) = z \operatorname{Im} z$

where x, y denote $\operatorname{Re} z, \operatorname{Im} z$ respectively.

Solution. We use u, v to denote the real part and imaginary part of f and x, y to denote the real and imaginary part of z.

a). We claim $f'(z)$ exists precisely at $z \neq 0$.
Note that for $z \neq 0$, $u(x,y) = \frac{x}{x^2+y^2}$ and $v(x,y) = \frac{-y}{x^2+y^2}$. This follows from consdering $f(z) = \frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{z}{|z|^2}$.
Then $u_x = \frac{y^2-x^2}{(x^2+y^2)^2} = v_y$ and $u_y = \frac{-2xy}{(x^2+y^2)^2} = -v_x$ for all $x, y \neq 0$. Moreover, u_x, u_y, v_x, v_y are continuous (smooth indeed) for all $x, y \neq 0$. Therefore f' is differentiable on $z \neq 0$,
on which, the derivative is given by $f'(z) = u_x + iv_x = \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{-2xyi}{(x^2+y^2)^2} = \frac{-\overline{z}^2}{z^2\overline{z}^2} = \frac{-1}{z^2}$
When $z = 0$, f does not even exist, so f' does not exist.

b). We claim $f'(z)$ exists preciesly at points with $x = y$.
Note that for all $x, y \in \mathbb{C}$, $u(x,y) = x^2, v(x,y) = y^2$. Hence, $u_x = 2x, u_y = 0, v_x = 0, v_y = 0$. Then the CR-equation is satisfied only when $x = y = 0$, so f is not differentiable except when $x = y$.
When $x = y$, the partial-derivatives are continuous. Moreover, from the above, partial-derivatives exist everywhere and hence exist in a neighborhood of the points. Together with the satisfaction of C-R equations, f is differentiable at the points, on which, $f'(z) = u_x + iv_x = 2x = 2\operatorname{Re}(z)$.

c). We claim $f'(z)$ exists preciesly at 0.
Note that for all $x, y \in \mathbb{C}$, $u(x,y) = xy, v(x,y) = y^2$. Hence, $u_x = y, u_y = x, v_x = 0, v_y = 2y$. Then the CR-equation is satisfied only when $x = y = 0$, so f is not differentiable except when $x = y = 0$.
When $x = y = 0$, the partial-derivatives are continuous. Moreover, from the above, partial-derivatives exist everywhere and hence exist in a neighborhood of the point. Together with the satisfaction of C-R equations, f is differentiable at the point, on which, $f'(0) = u_x + iv_x = 0$.

6 (P.89, Q3)**.** Using the Caucy-Riemann equations and the theorem in Sec. 21, show that the function $f(z) = exp(\overline{z})$ is not analytic anywhere on $\mathbb{C}$

Solution. Show that on some points, the C-R equations are not satisfied.

7 (P.89, Q4)**.** Let $f(z) = exp(z^2)$.

(i). Give two proofs that the function f is entire, that is, complex differentiable (or holomorphic) on all of $\mathbb{C}$.

(ii). Find the derivative of f.

Solution.

(i). First we use the C-R equations. Note that $f(z) = \exp\left(z^2\right) = \exp\left(x^2 - y^2\right)\exp(2xyi)$. Therefore, we have $u(x, y) = \exp\left(x^2 - y^2\right)\cos(2xy)$ and $v(x, y) = \exp\left(x^2 - y^2\right)\sin(2xy)$ for all $z \in \mathbb{C}$. Note that the CR equations are satisfied everywhere:

$$u_x = 2\exp\left(x^2 - y^2\right)(x\cos(2xy) - y\sin(2xy)) = v_y$$
$$u_y = -2\exp\left(x^2 - y^2\right)(y\cos(2xy) + x\sin(2xy)) = -v_x$$

for all $z \in \mathbb{C}$. In addition the partial derivatives exist and continuous everywhere. Hence, f is complex differential everywhere.

Second, we use the chain rule. Let $g(z) = z^2$ and $h(z) = \exp(z)$. Note that g, h are entire and $f(z) = h \circ g(z)$. For any $z_0 \in \mathbb{C}$, g is differentiable at z_0 and h is differentiable at $g(z_0)$. By chain rule, $h \circ g$ is differentiable at z_0. Hence $f = h \circ g$ is differentiable everywhere.

(ii). Using the notations in (i), $f'(z) = h'(g(z))g'(z) = 2z\exp\left(z^2\right)$ for all $z \in \mathbb{C}$ by chain rule.

8 (P.108, Q11)**.** *(Modified on 27 Sep).* Using the Cauchy-Riemann equations and the theorem in Sec. 21, show that neither of the functions $z \mapsto \sin(\overline{z})$ and $z \mapsto \cos(\overline{z})$ are holomorphic (complex differentiable) *everywhere* on $\mathbb{C}$, that is, the functions are not entire functions.

Solution. Show that on some points, the C-R equations are not satisfied.