MATH 2230A - HW 3*(Modified on 27 Sep)*
Due Date: 5 Oct 2020 (Mon), 23:59
(Please submit assignments to Blackboard and follow the instructions there.)

Problems: P.61-62: Q8, 9; P.70-71: Q1, 2, 3; P. 89: Q3, 4; P. 108: Q11. (8 Questions in total)

Textbook: Brown JW, Churchill RV(2014). Complex Variables and Applications, nineth edition, McGraw-Hill Education

We type here all the required problems *for your convenience only.* The presentation of the problems here may be different from the original one but the respective solution should be unaffected.

Remark. You can safely refer *the theorem in Sec. 21* and *the theorem in Sec. 23* below to Theorem 1.20 and Theorem 1.21 (P.19-20) of the Lecture Note respectively. The theorems are about characerizaing complex differentiability via Cauchy-Riemann Equations.
You can also safely refer *the Example 2 in Sec. 19* to Example 2 in P.17 of the Lecture note. This is about the differentiability of the conjugate function.
Please have a look at the respective section in the Lecture Note before attempting this HW.

1 (P.61-62 Q8)**.** Using the method in Example 2, Sec. 19, show that $f'(z)$ does not exist at any $z \in \mathbb{C}$ if

a) $f(z) = \mathrm{Re}(z)$

b) $f(z) = \mathrm{Im}(z)$

2 (P.61-62 Q9)**.** Let f be a function on $\mathbb{C}$ defined by

$$f(z) = \begin{cases} \frac{\overline{z}^2}{z} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

Consider f to map from the z-plane to the w- plane. Consider f at the point $z_0 = 0$. Denote $\Delta z := z - z_0$, and hence $\Delta w, \Delta x, \Delta y$ as usual

(i). Show that $\frac{\Delta w}{\Delta z} = 1$ at each nonzero point on the real and imaginary axes in the Δz plane.

(ii). Show that $\frac{\Delta w}{\Delta z} = -1$ on the line $\Delta y = \Delta x$ in the Δz plane.

(iii). Hence, show that $f'(0)$ does not exist.

3 (P.70-71 Q1)**.** Using the theorem in Sec. 21, show that $f'(z)$ does not exist at any point $z \in \mathbb{C}$ if f is defined by

a) $f(z) = \overline{z}$

b) $f(z) = z - \overline{z}$

c) $f(z) = 2x + ixy^2$

d) $f(z) = e^x e^{-iy}$

where x, y denote $\mathrm{Re}\, z, \mathrm{Im}\, z$ respectively.

4 (P.70-71, Q2)**.** Using the theorem in Sec. 23, for each of the following functions f on $\mathbb{C}$

(i). show that $f'(z)$ and its derivative $f''(z)$ exist everywhere on $\mathbb{C}$

(ii). find $f''(z)$.

a) $f(z) = iz + 2$

b) $f(z) = e^{-x}e^{-iy}$

c) $f(z) = z^3$

d) $f(z) = \cos x \cosh y - i \sin x \sinh y$

where x, y denote $\mathrm{Re}\, z, \mathrm{Im}\, z$ respectively.

5 (P.70-71, Q3)**.** Using theorems in Sec. 21 and 23, for each of the following functions f on $\mathbb{C}$, determine where $f'(z)$ exists and find the corresponding value for such z.

a) $f(z) = \frac{1}{z}$ b) $f(z) = x^2 + iy^2$ c) $f(z) = z \operatorname{Im} z$

where x, y denote $\operatorname{Re} z, \operatorname{Im} z$ respectively.

6 (P.89, Q3)**.** Using the Caucy-Riemann equations and the theorem in Sec. 21, show that the function $f(z) = exp(\overline{z})$ is not analytic anywhere on $\mathbb{C}$

7 (P.89, Q4)**.** Let $f(z) = exp(z^2)$.

(i). Give two proofs that the function f is entire, that is, complex differentiable (or holomorphic) on all of $\mathbb{C}$.

(ii). Find the derivative of f.

8 (P.108, Q11)**.** *(Modified on 27 Sep).* Using the Cauchy-Riemann equations and the theorem in Sec. 21, show that neither of the functions $z \mapsto \sin(\overline{z})$ and $z \mapsto \cos(\overline{z})$ are holomorphic (complex differentiable) *everywhere* on $\mathbb{C}$, that is, the functions are not entire functions.