13-3

$$\frac{Re(z_1+z_2)}{|z_3+z_4|} \leq \frac{|Re(z_1+z_2)|}{|z_3+z_4|} \leq \frac{|z_1+z_2|}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}$$

13-5, 13-6 Skipped.

23-1

(a) $z = \frac{-2}{1+\sqrt{3}i} = \frac{-2(1-\sqrt{3}i)}{4} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos(\frac{2}{3}\pi + 2n\pi) + \sin(\frac{2}{3}\pi + 2n\pi)\,i$

$\Rightarrow Arg\, z = \frac{2}{3}\pi$

(b) $z = (\sqrt{3}-i)^6 = (2(\frac{\sqrt{3}}{2} - \frac{1}{2}i))^6 = 64(e^{(-\frac{\pi}{6})i})^6 = \cancel{64 e^{(-\pi+2n\pi)i}}\ 64e^{-\pi i}$

$\Rightarrow Arg\, z = -\pi + 2\pi = \pi \quad (Arg\, z \in (-\pi, \pi])$.

23-2

(a) $|e^{i\theta}| = |\cos\theta + i\sin\theta| = 1$

(b) $\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos(-\theta) + i\sin(-\theta) = e^{-\theta i}$

23-4 skipped.

23-5 ((c) & (d) skipped)

(a) $i(1-\sqrt{3}i)(\sqrt{3}+i) = 4e^{\frac{\pi}{2}i}e^{-\frac{\pi}{3}i}e^{\frac{\pi}{6}i} = 4e^{\frac{\pi}{3}i} = 2(1+\sqrt{3}i)$

(b) $5i/(2+i) = \frac{5}{\sqrt{5}}\frac{e^{\frac{\pi}{2}i}}{e^{\theta i}} = \sqrt{5}e^{(\frac{\pi}{2}-\theta)i} = \sqrt{5}(\frac{\sqrt{5}}{5} + \frac{2\sqrt{5}}{5}i) = 1+2i$

($\theta = \arctan\frac{1}{2}$)

23-7

$(z^m)^{-1} = (r^m e^{im\theta})^{-1} = \frac{1}{r^m} e^{i(-m\theta)}$

$(z^{-1})^m = (r^{-1} e^{i\theta})^m = r^{-m} e^{-im\theta} = (z^m)^{-1}$.

23-9

Let $S_n = \sum_{k=1}^{n} z^n$, then $zS_n - S_n = z^{n+1} - 1 \Rightarrow S_n = \frac{1-z^{n+1}}{1-z}$.

As ~~z^n~~ if $z = e^{i\theta}$, $z^n = e^{in\theta} = \cos n\theta + i\sin n\theta$,

we can plug $z = e^{i\theta}$ into the sum and take the real part of both sides,

$\Rightarrow 1 + \cos\theta + \dots + \cos n\theta = \cancel{\mathrm{Re}\left(\frac{1-(\cos(n+1)\theta + \sin(n+1)\theta i)}{1-\cos\theta - \sin\theta i}\right)}$

$= \mathrm{Re}\left(\frac{1-e^{(n+1)\theta i}}{1-e^{\theta i}}\right)$

$= \mathrm{Re}\left(\frac{e^{-\frac{\theta}{2}i} - e^{(n+\frac{1}{2})\theta i}}{e^{-\frac{\theta}{2}i} - e^{\frac{\theta}{2}i}}\right)$

$= \mathrm{Re}\left(\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2} i - \cancel{e^{(n+\frac{1}{2})}} \cos(n+\frac{1}{2})\theta - \sin(n+\frac{1}{2})\theta i}{2i\sin\frac{\theta}{2}}\right)$

$= \frac{\sin\frac{\theta}{2} + \sin(n+\frac{1}{2})\theta i}{2\sin\frac{\theta}{2}}$