Mean Value Theorem

Definition. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a function.

- f is said to have an *absolute/global maximum* at $c \in I$ if $f(c) \geq f(x)$ for all $x \in I$.
- f is said to have an *absolute/global minimum* at $c \in I$ if $f(c) \leq f(x)$ for all $x \in I$.
- f is said to have a *relative/local maximum* at $c \in I$ if there exists $\delta > 0$ such that
$$f(c) \geq f(x), \quad \forall x \in I \cap (c - \delta, c + \delta).$$
- f is said to have a *relative/local minimum* at $c \in I$ if there exists $\delta > 0$ such that
$$f(c) \leq f(x), \quad \forall x \in I \cap (c - \delta, c + \delta).$$

Example 1. 0 is not a relative extremum point of the function $f : [0, \infty) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Proof. We need to show that for any $\delta > 0$, there exist $x_1, x_2 \in [0, \delta)$ such that

$$f(x_1) < f(0) < f(x_2).$$

Consider the sequences (u_n) and (v_n) in $[0, \infty)$ defined by

$$u_n = \frac{1}{2n\pi + \pi/2} \quad \text{and} \quad v_n = \frac{1}{2n\pi + 3\pi/2}, \quad \forall n \in \mathbb{N}.$$

Note that $\sin(1/u_n) = 1$ and $\sin(1/v_n) = -1$. Hence $f(v_n) < f(0) < f(u_n)$ for all $n \in \mathbb{N}$. Also, notice that both sequences converge to 0. Hence for any $\delta > 0$, there exists $N \in \mathbb{N}$ such that

$$u_N < \delta, \quad \text{and} \quad v_N < \delta \quad \implies \quad u_N, v_N \in [0, \delta).$$

It follows that we can take $x_1 = v_N$ and $x_2 = u_N$. □

The following observation is simple, but it is essential to develop important results.

Interior Extremum Theorem (c.f. 6.2.1)**.** *Let c be an interior point of the interval I at which $f : I \to \mathbb{R}$ has a relative extremum. If the derivative of f at c exists, then $f'(c) = 0$.*

From this result, we can deduce the following useful theorems:

Rolle's Theorem (c.f. 6.2.3)**.** *Suppose f is continuous on a closed interval $[a, b]$, that the derivative f' exists at every point of the open interval (a, b), and that $f(a) = f(b) = 0$. Then there exists at least one point c in (a, b) such that $f'(c) = 0$.*

Mean Value Theorem (c.f. 6.2.4)**.** *Suppose f is continuous on a closed interval $[a, b]$, that the derivative f' exists at every point of the open interval (a, b). Then there exists at least one point c in (a, b) such that*

$$f(b) - f(a) = f'(c)(b - a).$$

One application of the **Mean Value Theorem** is deducing inequalities.

Example 2 (c.f. Example 6.2.10(b))**.** Show that for any $x \geq 0$, we have $-x \leq \sin x \leq x$.

Solution. We need to divide the proof into two cases:

- Suppose $x = 0$. It is clear that the required inequality is indeed an equality.

- Suppse $x > 0$. Consider the sine function, which is continuous and differentiable on $\mathbb{R}$. In particular, it is continuous on $[0, x]$ and differentiable on $(0, x)$. Hence by the **Mean Value Theorem**, there exists $c \in (0, x)$ such that

  $$\sin x - \sin 0 = \cos c \cdot (x - 0) \quad \iff \quad \sin x = \cos c \cdot x.$$

  Since $-1 \leq \cos c \leq 1$ and $x > 0$, we have $-x \leq \sin x \leq x$.

Example 3. Show that for any $x > 0$, we have $\dfrac{x}{1+x} < \ln(1 + x) < x$.

Solution. Consider the function f given by $f(x) = \ln(1 + x)$. Note that f is is continuous and differentiable on $(-1, \infty)$. In particular, it is continuous on $[0, x]$ and differentiable on $(0, x)$. Hence by the **Mean Value Theorem**, there exists $c \in (0, x)$ such that

$$\ln(1 + x) - \ln(1 + 0) = \frac{1}{1 + c} \cdot (x - 0) \quad \iff \quad \ln(1 + x) = \frac{1}{1 + c} \cdot x.$$

Since $0 < c < x$, we have

$$\frac{1}{1 + x} < \frac{1}{1 + c} < \frac{1}{1 + 0} = 1.$$

It follows that

$$\frac{x}{1 + x} < \ln(1 + x) < x.$$

Another application of the **Mean Value Theorem** is approximation.

Example 4 (c.f. Example 6.2.9(b))**.** Correct $\sqrt{105}$ to 1 decimal place.

Solution. Consider the square root function, which is continuous on $[100, 105]$ and differentiable on $(100, 105)$. Hence by the **Mean Value Theorem**,

$$\sqrt{105} - \sqrt{100} = \frac{1}{2\sqrt{c}} \cdot (105 - 100), \quad \text{for some } c \in (100, 105).$$

Note that since $100 < c < 105 < 121$, we have $10 < \sqrt{c} < 11$. It follows that

$$\frac{5}{2} \cdot \frac{1}{11} < \frac{5}{2\sqrt{c}} < \frac{5}{2} \cdot \frac{1}{10} \quad \Longrightarrow \quad \frac{225}{22} < \sqrt{105} < \frac{41}{4}.$$

Since $225/22 \approx 10.227$ and $41/4 = 10.25$, we have $\sqrt{105} = 10.2$ correct to 1 decimal place.

Taylor's Theorem

Here are two generalizations of the **Mean Value Theorem**.

Cauchy Mean Value Theorem (c.f. 6.3.2)**.** *Let f and g be continuous on $[a,b]$ and differentiable on (a,b), and assume that $g'(x) \neq 0$ for all $x \in (a,b)$. Then there exists $c \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Taylor's Theorem (c.f. 6.4.1)**.** *Let f be a function such that f and its derivatives f', f'', ..., $f^{(n)}$ are continuous on $[a,b]$ and that $f^{(n+1)}$ exists on (a,b). If $x_0 \in [a,b]$, then for any $x \in [a,b]$ there exists a point c between x and x_0 such that*

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)(x_0)}}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

Remark. In this case, we denote respectively the n-th order **Taylor Polynomial** of f and the **Remainder Term** of f by:

$$\begin{aligned} P_n(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\ R_n(x) &= f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}. \end{aligned}$$

Example 5. Show that for any $x > 0$, we have $x - x^3/6 \leq \sin x \leq x + x^3/6$.

Solution. Notice that the function $f(x) = \sin x$ is infinitely differentiable on $\mathbb{R}$:

$$f^{(n)}(x) = \begin{cases} \sin x, & \text{if } n = 4k, \\ \cos x, & \text{if } n = 4k+1, \\ -\sin x, & \text{if } n = 4k+2, \\ -\cos x, & \text{if } n = 4k+3, \end{cases}$$

In particular, we have $f^{(n)}(0) = 0, 1, 0, -1, 0, 1, 0, -1, ...$ for $n = 0, 1, 2, 3, 4, 5, 6, 7, ...$ Fix any $n \in \mathbb{N}$ and apply the **Taylor's Theorem** with $x_0 = 0$ yields $c \in (0, x)$ such that

$$f(x) = f(0) + f'(0)(x-0) + \cdots + \frac{f^{(n)}(0)}{n!}(x-0)^n + \frac{f^{(n+1)}(c)}{n!}(x-0)^{n+1}. \tag{1}$$

In particular if we take $n = 2$,

$$\sin x = 0 + 1 \cdot x + 0 \cdot x^2 + \frac{-\cos c}{3!} \cdot x^3 = x - \frac{\cos c}{6}x^3.$$

Since $-1 \leq \cos c \leq 1$ and $x > 0$, we have

$$-\frac{x^3}{6} \leq \frac{\cos c}{6}x^3 \leq \frac{x^3}{6} \quad \Longrightarrow \quad x - \frac{x^3}{6} \leq \sin x \leq x + \frac{x^3}{6}.$$

Remark. Compare it with **Example 2**.

Example 6. Approximate $\sin(0.5)$ with error less than 10^{-5}.

Solution. We need to approximate the sine function with a suitable polynomial (so that we can compute by hand). The error of the polynomial to the sine function evaluated at $x = 0.5$ should not exceed 10^{-5}.
Using the **Taylor's Theorem**, we can approximate the value of $\sin(0.5)$ by the **Taylor Polynomial**. The error is controlled by the **Remainder Term**. By (1), there exists $c \in (0, 0.5)$ such that

$$|\sin(0.5) - P_n(0.5)| = |R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}(0.5-0)^{(n+1)} \leq \frac{1}{2^{n+1}} \cdot \frac{1}{(n+1)!}.$$

So the error is controlled by n. We require:

$$\frac{1}{2^{n+1}} \cdot \frac{1}{(n+1)!} < 10^{-5} \qquad \Longleftrightarrow \qquad 2^{n+1} \cdot (n+1)! > 10^5 = 100000.$$

Notice that

- If $n = 1$, $2^{n+1} \cdot (n+1)! = 8 < 100000$.
- If $n = 2$, $2^{n+1} \cdot (n+1)! = 48 < 100000$.
- If $n = 3$, $2^{n+1} \cdot (n+1)! = 384 < 100000$.
- If $n = 4$, $2^{n+1} \cdot (n+1)! = 3840 < 100000$.
- If $n = 5$, $2^{n+1} \cdot (n+1)! = 46080 < 100000$.
- If $n = 6$, $2^{n+1} \cdot (n+1)! = 645120 > 100000$.

Hence it suffices to approximate the value of $\sin(0.5)$ by $P_6(0.5)$. i.e.,

$$\begin{aligned}\sin(0.5) \approx P_6(0.5) &= 1 \cdot (0.5) + \frac{-1}{3!} \cdot (0.5)^3 + \frac{1}{5!}(0.5)^5 \\ &= \frac{1}{2} - \frac{1}{48} + \frac{1}{3840} \\ &= 0.47942708333...\end{aligned}$$