# Series of Real Numbers

**Definition** (c.f. Definition 3.7.1)**.** Let $(x_n)$ be a sequence of real numbers. Denote $s_n$ the *n-th partial sum* of the *series* $\sum x_n$, given by

$$s_n = x_1 + x_2 + \cdots + x_n = \sum_{k=1}^{n} x_k.$$

The series $\sum x_n$ is said to converge if $(s_n)$ converges. In this case, we denote

$$\sum_{k=1}^{\infty} x_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} x_k.$$

**Example 1** (c.f. Example 3.3.3(b))**.** The *harmonic series* $\sum 1/n$ is divergent.

*Proof.* Let $h_n$ denote the $n$-th partial sum of the harmonic series. Note that for each $n \in \mathbb{N}$,

$$\begin{aligned} h_{2^n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right) \\ &\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^n} + \cdots + \frac{1}{2^n}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = 1 + \frac{n}{2} \end{aligned}$$

It follows that $(h_n)$ is unbounded, hence it must be divergent. □

**Example 2** (c.f. Example 3.7.6(f))**.** The *alternating harmonic series* is convergent.

*Proof.* Let $s_n$ denote the $n$-th partial sum of the alternating harmonic series. Note that

$$\begin{aligned} s_{2n} &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right), \quad \forall n \in \mathbb{N} \\ s_{2n+1} &= 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \cdots - \left(\frac{1}{2n} - \frac{1}{2n+1}\right), \quad \forall n \in \mathbb{N} \end{aligned}$$

Thus $(s_{2n})$ is an increasing sequence and $(s_{2n+1})$ is a decreasing sequence such that

$$0 < s_{2n} < s_{2n+1} < 1, \quad \forall n \in \mathbb{N}.$$

By the **Monotone Convergence Theorem**, both of $(s_{2n})$ and $(s_{2n+1})$ are convergent. Moreover, they converge to the same value $\alpha \in \mathbb{R}$ because

$$s_{2n+1} = s_{2n} + \frac{1}{2n+1}, \quad \forall n \in \mathbb{N}.$$

Let $\varepsilon > 0$. By definition of limit, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$|s_{2n} - \alpha| < \varepsilon, \quad \forall n \geq N_1 \quad \text{and} \quad |s_{2n+1} - \alpha| < \varepsilon, \quad \forall n \geq N_2.$$

Take $N = \max\{2N_1, 2N_2 + 1\}$ and suppose $n \geq N$.

- If $n$ is even, then

$$n/2 \geq N/2 \geq N_1 \quad \Longrightarrow \quad |s_n - \alpha| = |s_{2(n/2)} - \alpha| < \varepsilon.$$

- If $n$ is odd, then

$$(n-1)/2 \geq (N-1)/2 \geq N_2 \quad \Longrightarrow \quad |s_n - \alpha| = |s_{2[(n-1)/2]+1} - \alpha| < \varepsilon.$$

In any cases, $|s_n - \alpha| < \varepsilon$. It follows that $(s_n)$ converges to $\alpha$. □

When we are given a series, we usually want to determine whether it is convergent or not. The following are some basic tests of convergence.

**The $n$-th term Test** (c.f. 3.7.3). *If the series $\sum x_n$ converges, then $\lim x_n = 0$.*

**Remark.** Its contrapositive statement is useful. i.e., if $(x_n)$ does not converge to 0, then the series $\sum x_n$ is divergent.

**Cauchy Criterion for Series** (c.f. 3.7.4). *The series $\sum x_n$ converges if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$|x_{n+1} + x_{n+2} + \cdots + x_{n+p}| < \varepsilon, \quad \forall n \geq N, \quad \forall p \in \mathbb{N}.$$

**Comparison Test** (c.f. 3.7.7). *Let $(x_n)$ and $(y_n)$ be sequences of real numbers. Suppose there exists $K \in \mathbb{N}$ such that*

$$0 \leq x_n \leq y_n, \quad \forall n \geq K.$$

*Then*

*(a) the convergence of $\sum y_n$ implies the convergence of $\sum x_n$.*

*(b) the divergence of $\sum x_n$ implies the divergence of $\sum y_n$.*

**Definition** (c.f. Definition 9.1.1). Let $(x_n)$ be a sequence of real numbers. The series $\sum x_n$ is said to *converge absolutely* if the series $\sum |x_n|$ is convergent. The series $\sum x_n$ is said to *converge conditionally* if it is convergent but not absolutely convergent.

**Example 3.** Every convergent series without negative terms is absolutely convergent. e.g.,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{2^n}.$$

From **Example 1 and 2**, the alternating harmonic series is conditionally convergent.

**Theorem** (c.f. Theorem 9.1.2). *A series must be convergent if it is absolutely convergent.*

**Rearrangement Theorem** (c.f. 9.1.5). *Let $\sum x_n$ be an absolutely convergent series. Then for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$, $\sum x_{\sigma(n)}$ is also convergent and*

$$\sum_{n=1}^{\infty} x_{\sigma(n)} = \sum_{n=1}^{\infty} x_n.$$

**Remark.** A series $\sum x_n$ is said to be *unconditionally convergent* if $\sum x_{\sigma(n)}$ converges to the same value for all bijection $\sigma : \mathbb{N} \to \mathbb{N}$.

# Tests of Absolute Convergence

The following tests of absolute convergence are mainly due to the **Comparison Test**.

**Root Test** (c.f. 9.2.2)**.** *Let $(x_n)$ be a sequence real numbers.*

*(a) If there exists $r < 1$ and $K \in \mathbb{N}$ such that*

$$|x_n|^{1/n} \leq r, \quad \forall n \geq K,$$

*then the series $\sum x_n$ is absolutely convergent.*

*(b) If there exists $K \in \mathbb{N}$ such that*

$$|x_n|^{1/n} \geq 1, \quad \forall n \geq K,$$

*then the series $\sum x_n$ is divergent.*

**Ratio Test** (c.f. 9.2.4)**.** *Let $(x_n)$ be a sequence of non-zero real numbers.*

*(a) If there exists $r < 1$ and $K \in \mathbb{N}$ such that*

$$\left|\frac{x_{n+1}}{x_n}\right| \leq r, \quad \forall n \geq K,$$

*then the series $\sum x_n$ is absolutely convergent.*

*(b) If there exists $K \in \mathbb{N}$ such that*

$$\left|\frac{x_{n+1}}{x_n}\right| \geq 1, \quad \forall n \geq K,$$

*then the series $\sum x_n$ is divergent.*

**Remark.** The **Root Test** and the **Ratio Test** are inconclusive when $r = 1$.

**Integral Test** (c.f. 9.2.6)**.** *Let $f : [1, \infty) \to \mathbb{R}$ be a continuous, decreasing, positive function. Then the series $\sum f(n)$ is convergent if and only if the improper integral*

$$\int_1^\infty f(x)dx = \lim_{b\to\infty} \int_1^b f(x)dx$$

*exists. In this case,*

$$\int_{N+1}^\infty f(x)dx \leq \sum_{n=N+1}^{\infty} f(n) \leq \int_N^\infty f(x)dx, \quad \forall N \in \mathbb{N}\,.$$

**Remark.** An application of the **Integral Test** implies that the $p$-series $\sum 1/n^p$ is convergent for $p > 1$ and divergent for $p \leq 1$.

**Example 4** (c.f. Section 9.2, Ex.2, 3, 4 & 7)**.** Determine the convergence of the following series.

(a) $\sum_{n=1}^{\infty} n^n e^{-n}$

(b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(c) $\sum_{n=2}^{\infty} (\ln n)^{-\ln n}$

(d) $\sum_{n=2}^{\infty} (n \ln n)^{-1}$

(e) $\sum_{n=1}^{\infty} n! e^{-n^2}$

(f) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

**Solution.** Let's check the convergence of the series using suitable tests.

(a) We apply the **Root Test** here. Note that

$$|x_n|^{1/n} = |n^n e^{-n}|^{1/n} = \frac{n}{e} \geq 1, \quad \forall n \geq 3.$$

Hence the series is **divergent**.

(b) We apply the **Ratio Test** here. Note that

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n}.$$

Therefore we have

$$\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-n} = \frac{1}{e} < 1.$$

Hence the series is **convergent**.

(c) We apply the **Comparison Test** here. Note that

$$\ln(x_n) = -\ln n \ln(\ln n) \leq -2\ln n, \quad \forall n \geq 2000.$$

(Here we want $\ln(\ln n) \geq 2$. i.e., $n \geq e^{e^2} \approx 1618.17$.) Hence we have

$$0 \leq x_n \leq \frac{1}{n^2}, \quad \forall n \geq 2000.$$

Since $\sum 1/n^2$ is convergent, the series is also **convergent**.

(d) We apply the **Integral Test** here. Consider the function $f : [2, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x \ln x}.$$

Then $f$ is a continuous, decreasing, positive function with $f(n) = x_n$. Also, if the improper integral exists, it is given by

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{b\to\infty} \int_2^b \frac{1}{\ln x} d(\ln x) = \lim_{b\to\infty} \Big[\ln(\ln x)\Big]_{x=2}^{x=b} = \infty.$$

Hence the improper integral does not exist, therefore the series is **divergent**.

(e) We apply the **Ratio Test** here. Note that

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{(n+1)!e^{-(n+1)^2}}{n!e^{-n^2}} = \frac{(n+1)!}{n!} \cdot \frac{e^{n^2}}{e^{(n+1)^2}} = \frac{n+1}{e^{2n+1}}.$$

Applying the **L'Hospital's Rule**, we have

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty}\frac{n+1}{e^{2n+1}} = \lim_{n\to\infty}\frac{1}{2e^{2n+1}} = 0 < 1.$$

Hence the series is **convergent**.

(f) We apply the $n$**-th Term Test** here. Note that

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty}\frac{(-1)^{2n}\cdot 2n}{2n+1} = 1 \neq 0.$$

Since the subsequence $(x_{2n})$ of $(x_n)$ does not converge to 0, $(x_n)$ must not converge to 0. Hence the series is **divergent**.