THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 10 (April 8)

Theorem 1. *Let $A \subseteq \mathbb{R}$, let $f\colon A \to \mathbb{R}$, and let $f(x) \geq 0$ for all $x \in A$. We let $\sqrt{f}$ be defined for $x \in A$ by $(\sqrt{f})(x) := \sqrt{f(x)}$.*

(a) If f is continuous at a point $c \in A$, then $\sqrt{f}$ is continuous at c.

(b) If f is continuous on A, then $\sqrt{f}$ is continuous on A.

Proof. (a) Note that, for $x \in A$,

$$\left|\sqrt{f(x)} - \sqrt{f(c)}\right|^2 \leq \left|\sqrt{f(x)} - \sqrt{f(c)}\right| \cdot \left|\sqrt{f(x)} + \sqrt{f(c)}\right| = \left|\sqrt{f(x)}^2 - \sqrt{f(c)}^2\right|,$$

so that

$$|\sqrt{f(x)} - \sqrt{f(c)}| \leq \sqrt{|f(x) - f(c)|}.$$

Let $\varepsilon > 0$. Since f is continuous at c, there is $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon^2 \quad \text{whenever } x \in A \cap V_\delta(c).$$

Now, if $x \in A \cap V_\delta(c)$, then

$$|(\sqrt{f})(x) - (\sqrt{f})(c)| \leq \sqrt{|f(x) - f(c)|} < \sqrt{\varepsilon^2} = \varepsilon.$$

Hence, $\sqrt{f}$ is continuous at c.

(b) It follows immediately from (a).

□

Theorem 2. *Let $A, B \subseteq \mathbb{R}$ and let $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ be functions such that $f(A) \subseteq B$. If f is continuous at a point $c \in A$ and g is continuous at $b = f(c) \in B$, then the composition $g \circ f\colon A \to \mathbb{R}$ is continuous at c.*

Theorem 3. *Let $A, B \subseteq \mathbb{R}$ and let $f\colon A \to \mathbb{R}$ be continuous on A, and let $g\colon B \to \mathbb{R}$ be continuous on B. If $f(A) \subseteq B$, then the composition $g \circ f\colon A \to \mathbb{R}$ is continuous on A.*

Example 1. (a) If $f\colon A \to \mathbb{R}$ is continuous on A, then $|f|$ is continuous on A.

(b) If $f\colon A \to \mathbb{R}$ is continuous on A and $f(x) \geq 0$ for $x \in A$, then $\sqrt{f}$ is continuous on A.

(c) If $f\colon A \to \mathbb{R}$ is continuous on A, then $\sin(f(x))$ is continuous on A.

Example 2. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be **additive** if $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Prove that if f is continuous at some point x_0, then it is continuous at every point of $\mathbb{R}$.

Solution. Let $c \in \mathbb{R}$. Note that, for $x \in \mathbb{R}$,

$$f(x) - f(c) = f(x-c) = f(x-c+x_0) - f(x_0).$$

Since f is continuous at x_0, there exists $\delta > 0$ such that

$$|f(y) - f(x_0)| < \varepsilon \quad \text{whenever } y \in V_\delta(x_0).$$

Now, if $x \in V_\delta(c)$, then $x - c + x_0 \in V_\delta(x_0)$, and hence

$$|f(x) - f(c)| = |f(x-c+x_0) - f(x_0)| < \varepsilon.$$

Therefore f is continuous at c. ◀

Classwork

1. Give an example of functions f and g that are both discontinuous at a point c in $\mathbb{R}$ such that

   (a) the sum $f + g$ is continuous at c,

   (b) the product fg is continuous at c.

   Solution. Take $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$ and $g(x) = 1 - f(x)$. ◀

2. Let f, g be continuous from $\mathbb{R}$ to $\mathbb{R}$, and suppose that $f(r) = g(r)$ for all rational numbers r. Is it true that $f(x) = g(x)$ for all $x \in \mathbb{R}$?

   Solution. Let $x \in \mathbb{R}$. The Density Theorem implies that for each $n \in \mathbb{N}$, there is $r_n \in \mathbb{Q}$ such that $x < r_n < x + 1/n$. In particular, $\lim(r_n) = x$. By the Sequential Criterion for Continuity, $f(x) = \lim(f(r_n))$ and $g(x) = \lim(g(r_n))$. Since $f(r_n) = g(r_n)$ for all $n \in \mathbb{N}$, it follows that $f(x) = g(x)$. ◀

3. Let $g\colon \mathbb{R} \to \mathbb{R}$ satisfy the relation $g(x+y) = g(x)g(y)$ for all $x, y \in \mathbb{R}$. Show that if g is continuous at $x = 0$, then g is continuous at every point of $\mathbb{R}$. Also if we have $g(a) = 0$ for some $a \in \mathbb{R}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.

   Solution. Note that $g(x) - g(c) = g(c)(g(x-c) - g(0))$ for any $x, c \in \mathbb{R}$. Let $c \in \mathbb{R}$, and let (x_n) be a sequence in $\mathbb{R}$ that converges to c. So $\lim(x_n - c) = 0$. Since g is continuous at 0, we have $\lim(g(x_n - c)) = g(0)$, and hence $\lim(g(x_n)) = g(c)$. By the Sequential Criterion for Continuity, g is continuous at c.

   If $g(a) = 0$, then

   $$g(x) = g(x-a)g(a) = 0 \quad \text{for all } x \in \mathbb{R}.$$

   ◀