MATH 2050C Mathematical Analysis I
2020-21 Term 2

Solution to Problem Set 11

5.3-2

Suppose $\lim x_n = x_0$. Then by the definition of continuity of f, g, we have

$$\lim f(x_n) = f(x_0), \quad \lim g(x_n) = g(x_n)$$

Note by $x_n \in E$, we have $f(x_n) = g(x_n)$, hence

$$\lim f(x_n) = \lim g(x_n)$$

So we have $f(x_0) = g(x_0)$, which implies $x_0 \in E$.

5.3-3

By the Maximum-Minimum Theorem 5.3.4, we can find $x_1, x_2 \in [a, b]$ with $f(x_1) = \max f(I), f(x_2) = \min f(I)$. We claim $f(x_1) \geq 0$ and $f(x_2) \leq 0$.

Now if we suppose $f(x_1) < 0$, then since $f(x_1)$ is a supremum of $f(I)$, for any $x \in [a, b]$, we have $f(x) \leq f(x_1)$. By the statement of problem, we can find $y \in [a, b]$ with $|f(y)| \leq \frac{1}{2}|f(x_1)|$. At least, we will have

$$-f(y) \leq \frac{1}{2}(-f(x_1)) \implies \frac{1}{2}f(x_1) \leq f(y) \implies f(x_1) < \frac{1}{2}f(x_1) \leq f(y)$$

which contradicts with the facts that $f(x_1)$ is an upper bound of $f(I)$.

Similarly, if we suppose $f(x_2) > 0$, we can find y with $\frac{1}{2}f(x_2) \geq f(y)$, which will implies $f(x_2) > f(y)$, which leads a contradiction with the fact $f(x_2)$ is a lower bound for $f(I)$.

So in summary, we have $f(x_1) \geq 0, f(x_2) \leq 0$. Hence, by Intermediate Value Theorem, we can find $c \in [x_1, x_2]$ or $c \in [x_2, x_1]$(depending on which is bigger) such that $f(c) = 0$.

5.3-6

Define $g(x) = f(x) - f(x + \frac{1}{2})$. There are three cases for the value of $g(0)$.

- If $g(0) = 0$, then $c = 0 \in [0, \frac{1}{2}]$ is a solution of $f(c) = f(c + \frac{1}{2})$.
- If $g(0) > 0$, then $g(\frac{1}{2}) = f(\frac{1}{2}) - f(1) = f(\frac{1}{2}) - f(0) = -g(0) < 0$. Hence by Intermediate Value Theorem, we can find $c \in [0, \frac{1}{2}]$ such that $g(c) = 0$. This means $f(c) = f(c + \frac{1}{2})$.
- If $g(0) < 0$, we have the similar proof to get $g(\frac{1}{2}) > 0$ and apply Intermediate Value Theorem to get $g(c) = 0$ for $c \in [0, \frac{1}{2}]$ and hence $f(c) = f(c+\frac{1}{2})$.

If we view the earth's equator as the unit circle $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 = 1\}$, the temperature function h is a continuous map from $\mathbb{S}^1$ to $\mathbb{R}$. On the other hand, $\mathbb{S}^1$ can be viewed as a interval $[0, 1]$ by identify their end point. For example, let $\varphi : [0, 1] \to \mathbb{S}^1$ defined by

$$\varphi(t) = (\cos(2\pi t), \sin(2\pi t)).$$

Then $\varphi(0) = \varphi(1)$. So the function $f(t) = h(\varphi(t))$ is a continuous function from $[0, 1]$ to $\mathbb{R}$ with $f(0) = f(1)$. So we can find $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$. This means

$$\begin{aligned} h(\cos(2\pi c), \sin(2\pi c)) &= h(\cos(2\pi(c + \frac{1}{2})), \sin(2\pi(c + \frac{1}{2}))) \\ &= h(-\cos(2\pi c), -\sin(2\pi c)) \end{aligned}$$

which means h has the same values at one pair of antipodal points on $\mathbb{S}^1$.

5.3-17

We proof the following claim first.

If $f : [0, 1] \to \mathbb{R}$ is continous and has two different values a, b with $a < b$, then f cannot has only rational (or irrational) values.

Indeed, suppose $f(x_1) = a, f(x_2) = b$, then for any $c \in (a, b)$, by Intermediate Value Theorem, we can find $x_0 \in [x_1, x_2]$ or $[x_2, x_1]$ such that $f(x_0) = c$. By Density Theorem, we can always find a rational number in (a, b), and we can also find a irrational number in (a, b). So this means f has to take rational values and irrational values. So the claim is proved.

So from this claim, we can see that if f is not a constant, f will take at least two different values and we can write these two values as a, b with $a < b$ and hence f has to take both rational and irrational values.

Hence if f has only rational (irrational) values, f has to be a constant.