MATH 2050C Mathematical Analysis I
2020-21 Term 2

Solution to Problem Set 10

4.2-4

Denote $f(x) := \cos\frac{1}{x}, x \in \mathbb{R} \setminus \{0\}$, $a_n = \frac{1}{2n\pi}, n \in \mathbb{N}$ and $b_n = \frac{1}{(2n+\frac{1}{2})\pi}, n \in \mathbb{N}$. Note that $a_n, b_n \neq 0, \forall n \in \mathbb{N}$ and that (a_n) and (b_n) are convergent sequences with common limit 0. Suppose that $\lim_{x\to 0} f(x) = L$ exist, which implies that $L = \lim(f(a_n)) = \lim(f(b_n))$ by Theorem 4.1.8(b) and Theorem 4.1.5. But $f(a_n) = \cos 2n\pi = 1$ and $f(b_n) = \cos(2n + \frac{1}{2})\pi = 0$ for any $n \in \mathbb{N}$. Thus $\lim f(a_n) = 1$ while $\lim f(b_n) = 0$, a contradiction.
Denote $g(x) := x\cos\frac{1}{x}, x \in \mathbb{R} \setminus \{0\}$. Note that $|\cos y| \leq 1, \forall y \in \mathbb{R}$. Given $\varepsilon > 0$, set $\delta = \varepsilon$. For any x satisfying $0 < |x| < \delta$,

$$|g(x)| = |x|\,|\cos(1/x)| \leq |x| < \varepsilon.$$

Since ε is arbitrary, $\lim_{x\to 0} x\cos\frac{1}{x} = 0$.

4.2-5

By the supposition, there exists $\delta_1 > 0$ and $M > 0$ so that $|f(x)| < M, \forall x \in (c - \delta_1, c + \delta_1)$. Given $\varepsilon > 0$, there exists $\delta_2 > 0$ so that $|g(x)| < \varepsilon/M, \forall x \in (c - \delta_2, c + \delta_2) \setminus \{c\}$. Set $\delta = \min\{\delta_1, \delta_2\}$. For x satisfying $0 < |x - c| < \delta$, we have

$$|f(x)g(x)| < M \cdot (\varepsilon/M) = \varepsilon.$$

Hence $\lim_{x\to c} fg = 0$.

4.2-9

(a). By Theorem 4.2.4(a), we know

$$\lim_{x\to c} g = \lim_{x\to c}(f + g) - g = \lim_{x\to c}(f + g) - \lim_{x\to c} g$$

So the limit $\lim_{x\to c} g$ exists.

(b). $\lim_{x\to g}$ might not exist. For example, let $f(x) = x$ and $g(x) = \frac{|x|}{x}$ when $x \neq 0$ and define $g(0) = 0$.

Clearly $\lim_{x\to 0} f = 0$ and $\lim_{x\to 0} fg = \lim_{x\to 0} |x| = 0$ but $\lim_{x\to 0} g = \lim_{x\to 0} \frac{|x|}{x}$ does not exist.

4.2-15

We consider the follwing two cases.

First, if $\lim_{x\to c} f = 0$, we just need to proof $\lim_{x\to c} \sqrt{f} = 0$. Indeed, by the definition of limit $\lim_{x\to c} f = 0$, for any $\epsilon > 0$, we can choose $\delta > 0$ such that if $0 < |x - c| < \delta$, then

$$|f(x)| < \epsilon^2$$

Then we have $|\sqrt{f(x)}| < \epsilon$. Hence $\lim_{x\to c} \sqrt{f} = 0$.

Second, if $\lim_{x\to c} f > 0$, we write $A = \lim_{x\to c} f$. Again, for any $\epsilon > 0$, we can choose $\delta > 0$, and if $0 < |x - c| < \delta$, then we have

$$|f(x) - A| < \epsilon\sqrt{A}$$

Then we get for any $0 < |x - c| < \delta$,

$$|\sqrt{f}(x) - \sqrt{A}| = \frac{|f(x) - \sqrt{A}|}{\sqrt{f(x)} + \sqrt{A}} < \frac{\epsilon\sqrt{A}}{\sqrt{A}} = \epsilon$$

Hence, $\lim_{x\to c} \sqrt{f} = \sqrt{A} = \sqrt{\lim_{x\to c} f}$.

5.1-3

We divided $x_0 \in [a, c]$ in three cases.

First, $x_0 \in [a, b)$, For any $\epsilon > 0$, we can find $\delta > 0$, such that if $0 < |x - x_0| < \delta$ with $x \in [a, b]$, then $|f(x) - f(x_0)| < \epsilon$ by the definition of continuous. Then we can choose $\delta' = \min\{\delta, (b - x_0)\}$, then for any $x \in V_{\delta'}(x_0) \cap [a, c]$, we will have $x \in [a, b]$ and $x \in V_\delta(x_0)$, then we will have $|h(x) - h(x_0)| = |f(x) - f(x_0)| < \epsilon$. Hence $\lim_{x\to x_0} h(x) = h(x_0)$.

Second, $x_0 \in (b, c]$. The steps are essentially same with above and we can get $\lim_{x\to x_0} h(x) = h(x_0)$.

At last, if $x_0 = b$, we need to combine the continuouty of f and g. For any $\epsilon > 0$, we can choose δ_1, δ_2 such that for $x \in V_{\delta_1}(b) \cap [a, b]$, we have $|f(x) - f(b)| < \epsilon$, and for $x \in V_{\delta_2}(b) \cap [b, c]$, we have $|g(x) - g(b)| < \epsilon$. Hence we can choose $\delta = \min\{\delta_1, \delta_2\}$, and for any $x \in V_\delta(b) \cap [a, c]$, we will have either $x \in V_\delta(b) \cap [a, b]$ or $x \in V_\delta(b) \cap [b, c]$, which will imply either $|h(x) - h(b)| = |f(x) - f(b)| < \epsilon$ or $|h(x) - h(b)| = |g(x) - g(b)| < \epsilon$. So we will get $|h(x) - h(b)| < \epsilon$ for all $x \in V_\delta(b)$. In conclusion, we have $\lim_{x\to x_0} h(x) = h(x_0)$ for all $x_0 \in [a, c]$ and hence $h(x)$ is continuous on $[a, c]$.

5.1-5

For $x \neq 2$,

$$f(x) = \frac{x^2 + x + 6}{x - 2} = \frac{(x-2)(x+3)}{x-2} = x + 3.$$

To hold the continuity at $x = 2$, by the Remark 1 of Theorem 5.1.2, the necessary and sufficient condition is that $f(2) = \lim_{x \to 2} f(x) = 5$. We conclude that $f(x)$ is continuous at $x = 2$ if and only if define $f(2) = 5$.

5.1-12

Fix $x \in \mathbb{R}$. By The Density Theorem, there exists (x_n) so that

$$x_n \in \mathbb{Q} \cap (x + \frac{1}{n+1}, x + \frac{1}{n}), \quad \forall n \in \mathbb{N}.$$

Thus $\lim(x_n) = x$. Since f is continuous and $f(x_n) = 0, \forall n \in \mathbb{N}$,

$$f(x) = \lim f(x_n) = 0.$$

Since x is arbitrary, we have $f(x) = 0, \forall x \in \mathbb{R}$.

5.2-6

For any $\epsilon > 0$, since g is continuous at b, we can find $\delta > 0$ such that for all $|x - b| < \delta$, we have $|g(x) - g(b)| < \epsilon$. And since $\lim_{x \to c} f = b$, so for $\epsilon' = \delta$, we can find $\delta' > 0$ such that if $0 < |x - c| < \delta'$, we have $|f(x) - b| < \delta$. Now by $|f(x) - b| < \delta$, we have $|g(f(x)) - g(b)| < \epsilon$. In other word, we will have

$$\lim_{x \to c} g(f(x)) = g(b)$$

5.2-11

By the continuous of f, g, we have

$$\lim(f(s_n)) = \lim_{x \to s} f(x) f(s), \lim(g(s_n)) = \lim_{x \to s} g(x) = g(s)$$

Note that $f(s_n) \geq g(s_n)$ since $s_n \in S$, then take limit $n \to \infty$, we have

$$f(s) = \lim f(s_n) \geq \lim g(s_n) = g(s)$$

which implies $s \in S$.