MATH 2050C Mathematical Analysis I
2020-21 Term 2

Solution to Problem Set 7

3.4-6

(a). First, we have

$$(n+1)^{\frac{1}{n+1}} < n^{\frac{1}{n}} \Leftrightarrow (n+1)^n < n^{n+1} \Leftrightarrow \left(\frac{n+1}{n}\right)^n < n$$

So by Example 3.3.6, we know that $\left(1+\frac{1}{n}\right)^n$ is bounded above by 3 strictly and hence $\left(1+\frac{1}{n}\right)^n < n$ is valid for $n \geq 3$. So for $n \geq 3$, we know that the sequence (x_n) is indeed decreasing and hence the limit exists. So we can assume $x = \lim x_n$.
(b). Indeed, we have

$$x = \lim x_{2n} = \lim(2^{\frac{1}{2n}} n^{\frac{1}{2n}}) = \lim 2^{\frac{1}{2n}} \times \left(\lim n^{\frac{1}{n}}\right)^{\frac{1}{2}} = 1 \times x^{\frac{1}{2}}$$

So we get $x = x^2 \implies x = 0$ or $x = 1$. But $x = 0$ cannot happen since $x_n \geq 1$ all the time. So we get $\lim n^{\frac{1}{n}} = 1$.

3.4-7(c)

First note $\left(\left(1+\frac{1}{n^2}\right)^{n^2}\right)$ is indeed a subsequence of $\left(\left(1+\frac{1}{n}\right)^n\right)$ with $n_k = k^2$. So they have the same limit as

$$\lim\left(\left(1+\frac{1}{n^2}\right)^{n^2}\right) = \lim\left(\left(1+\frac{1}{n}\right)^n\right) = e$$

Hence

$$\lim\left(\left(1+\frac{1}{n^2}\right)^{2n^2}\right) = \left(\lim\left(1+\frac{1}{n^2}\right)^{n^2}\right)^2 = e^2$$

3.4-14

We will try to find this subsequence by induction. First, we can find a natural number n_1 with $s-1 < x_{n_1} \leq s$ by the definition of supremum. Now suppose we have find $n_1 < \cdots < n_k$ such that $s - \frac{1}{l} < x_{n_l} \leq s$ for all $1 \leq l \leq k$, since $s \notin \{x_n : n \in \mathbb{N}\}$, we know $M = \max\{s - \frac{1}{k+1}, x_1, x_2, \ldots, x_{n_k}\} < s$. Hence by the definition of supremum, we can find n_{k+1} with $x_{n_{k+1}} > M$. Clearly we have $n_{k+1} > n_k$ by the choice of M. Then by Mathematical induction, we can find $n_1 < n_2 < \ldots$ such that $s - \frac{1}{k} < x_{n_k} \leq s$. This will clearly show that this subsequence satisfies

$$\lim_{k \to \infty} x_{n_k} = s$$

3.4-19

Denote $X_k = \sup\{x_n : n \geq k\}, Y_k = \sup\{y_n : n \geq k\}$ and $Z_k = \sup\{x_n + y_n : n \geq k\}$. From the definition of supremum, for all $n \geq k$, $x_n \leq X_k$, $y_n \leq Y_k$ and $x_n + y_n \leq X_k + Y_k$. Thus $X_k + Y_k$ is an upper bound of $\{x_n + y_n, n \geq k\}$. Since Z_k is the supremum of $\{x_n + y_n, n \geq k\}$,

$$Z_k \leq X_k + Y_k, \quad \forall k \in \mathbb{N}.$$

From Theorem 3.4.11(c), $(X_k), (Y_k)$ and (Z_k) are convergent and $\lim(X_k) = \limsup x_n$, $\lim(Y_k) = \limsup y_n$ and $\lim(Z_k) = \limsup(x_n + y_n)$. Theorem 3.2.5 told that for two CONVERGENT sequences (a_n) and (b_n) so that $a_n \leq b_n, \forall n \in \mathbb{N}$, we have $\lim(a_n) \leq \lim(b_n)$. Replacing $a_k = Z_k$ and $b_k = X_k + Y_k$, we have

$$\lim(Z_k) \leq \lim(X_k + Y_k) = \lim(X_k) + \lim(Y_k),$$

i.e.

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n.$$

Example which two sides are not equal. Let $x_n = (-1)^n, y_n = (-1)^{n+1}$. So $x_n + y_n = 0$. Hence

$$\limsup(x_n + y_n) = 0 < 2 = \limsup(x_n) + \limsup(y_n)$$