MATH 2050C Mathematical Analysis I
2020-21 Term 2

Solution to Problem Set 3

2.4-4(a)

For the infimum part, we show that $a \inf S$ is a lower bound of aS and $u \leq a \inf S$ for any lower bound u of aS. Since $\inf S \leq s, \forall s \in S$ and $a > 0$, $a \inf S \leq as, \forall as \in aS$. Thus $a \inf S$ is a lower bound of aS. Suppose u is a lower bound of aS, i.e. $u \leq as, \forall as \in aS$. Thus $u/a \leq s, \forall s \in S$. u/a is a lower bound of S and $u/a \leq \inf S$. We have $u \leq a \inf S$. As u is arbitrary lower bound, it follows that $a \inf S = \inf(aS)$ by the definition.
For the supremum part, the same idea as above.

2.4-6

Let $S = \{f(x) : x \in X\}$. From Example 2.4.1(a), we know

$$\sup(a + S) = a + \sup S = a + \sup\{f(x) : x \in X\}.$$

Now we need to show $a + S = \{a + f(x) : x \in X\}$. This is not hard and we will give a short proof. By definition, for $y \in a + S$, we have $y = a + s$ for some $s \in S$. Then by definition of S, we can find $x_0 \in X$ such that $s = f(x)$. Hence $y = a + f(x_0) \in \{a + f(x) : x \in X\}$. So $a + S \subset \{a + f(x) : x \in X\}$. Conversely, for any $a + f(x_0) \in \{a + f(x) : x \in X\}$, since $f(x_0) \in S$, we have $a + f(x_0) \in a + S$. Hence, $\{a + f(x) : x \in X\} \subset a + S$. So we've showed $\{a + f(x) : x \in X\} = a + S$.

Then

$$\sup\{a + f(x) : x \in X\} = \sup(a + S) = a + \sup\{f(x) : x \in X\}.$$

For the infimum part, we have the similar proof. As one need to show

$$\inf(a + S) = a + \inf S$$

following the argument in Example 2.4.1(a) and use the result $\{a + f(x) : x \in X\} = a + \{f(x) : x \in X\}$ to get the similar result.

2.4-11

Suppose $A = \{g(y) : y \in Y\}$, $B = \{f(x) : x \in X\}$. We will prove for any $a \in A, b \in B$, we have $a \leq b$.

Indeed, for any $a = g(y_0) \in A$ for some $y_0 \in Y$, $b = f(x_0) \in B$ for some $x_0 \in X$, we have $a = \inf\{h(x, y_0) : x \in X\}$ by definition. Note that $h(x_0, y_0)$ is an element in the set $\{h(x, y_0) : x \in X\}$, we will have

$$\inf\{h(x, y_0) : x \in X\} \leq h(x_0, y_0)$$

Similarly, we will have

$$f(x_0) = \sup\{h(x_0, y) : y \in Y\} \geq h(x_0, y_0)$$

Based on these two formulas, we have $a = g(y_0) \leq h(x_0, y_0) \leq f(x_0) = b$. Then we can apply the result of **Example 2.4.1 (b)** to get

$$\sup A \leq \inf B$$

which is exactly

$$\sup\{g(y) : y \in Y\} \leq \inf\{f(x) : x \in X\}$$

2.4-19

This is a direct consequence of The Density Theorem 2.4.8.

Since $u > 0$, we have $\frac{x}{u} < \frac{y}{u}$. Then by The Density Theorem 2.4.8, there exists a rational number $r \in \mathbb{Q}$ such that $\frac{x}{u} < r < \frac{y}{u}$. So $x < ru < y$.

2.5-7

Suppose $x \in \cap_{n=1}^{\infty} I_n$. Thus $x \in I_n, \forall n$ and $x \geq 0$. If $x > 0$, by Archimedean property, there exists some $N \in \mathbb{N}$ satisfying $Nx > 1$. Thus $x > \frac{1}{N}$ and $x \notin J_N$. Contradiction. So we have $x = 0$. Hence $\cap_{n=1}^{\infty} I_n = \{0\}$.