MATH 2050C Mathematical Analysis I
2020-21 Term 2

Solution to Problem Set 1

Notation: Ex. = Exercise; A.P. = Algebraic Properties of $\mathbb{R}$.

2.1-2

a) Note $(-a)+(-b)+(a+b) \xlongequal{\text{A.P.(A2)}} (-a)+(-b)+a+b \xlongequal{\text{A.P.(A1)}} (-a)+a+(-b)+b \xlongequal{\text{A.P.(A4,A2)}} 0+0 \xlongequal{\text{A.P.(A3)}} 0$.

Then by A.P.(A4), $-(a+b)=(-a)+(-b)$.

b) $(-a)\cdot(-b) \xlongequal{\text{Ex.1c}} ((-1)\cdot a)\cdot((-1)\cdot b) \xlongequal{\text{A.P.(M2)}} (-1)\cdot(-1)\cdot a\cdot b \xlongequal{\text{Ex.1d}} a\cdot b$.

c) If $a\neq 0$, $(1/(-a))\cdot(-a)=1$ and $-(1/a)\cdot(-a) \xlongequal{\text{Ex.1c}} -(1/a)\cdot(-1)\cdot a \xlongequal{\text{Ex.1d}} (1/a)\cdot a = 1$. By Theorem 2.1.3(a), $1/(-a)=-(1/a)$.

d) $-(a/b)+(a/b)=0$ and $(-a)/b+a/b \xlongequal{\text{A.P.(D)}} ((-a)+a)\cdot(1/b)=0$. By Exercise 1(a), $-(a/b)=(-a)/b$.

2.1-8

(a). Note that for $b\neq 0, c\neq 0$, we have $a/b=(ac)/(bc)$. (This is a consequence of A.P.(M1,M2,M4))

Since x,y are rational number, we can find $p_1,p_2,q_1,q_2\in\mathbb{Z}$ with $q_1\neq 0, q_2\neq 0$ such that $x=\frac{p_1}{q_1}, y=\frac{p_2}{q_2}$. So

$$x+y=\frac{p_1q_2}{q_1q_2}+\frac{p_2q_1}{q_1q_2} \xlongequal{\text{A.P.(M4,D)}} \frac{p_1q_2+p_2q_1}{q_1q_2}, xy \xlongequal{\text{A.P.(M2,M4)}} \frac{p_1p_2}{q_1q_2}$$

Hence, $x+y, xy$ are both rational numbers.

(b). Suppose x is a rational number and y is an irrational number. If on the contrary, $z:=x+y$ is a rational number, we know that $-x$ is a rational number and hence $z-x=z+(-x)$ is a rational number. This contradicts with the fact that $y=z-x$ is irrational.

Further more, if $x\neq 0$, we still assume $z:=xy$ is rational. Then by result of (a), we know $y=z/x$ is rational. This is a contradiction. Hence xy is irrational.

2.1-11

a) Clearly, we will have $1/a \neq 0$. If not, we have $1 = a \cdot (1/a) = a \cdot 0 = 0$, a contradiction. By Theorem 2.1.8(a), we know $(1/a)^2 > 0$. Hence $a \cdot (1/a)^2 > 0$, which implies $1/a > 0$. $1/(1/a) \xlongequal{(M3)(M4)} (a \cdot (1/a)) \cdot (1/(1/a)) \xlongequal{(M2)} a \cdot ((1/a) \cdot (1/(1/a))) \xlongequal{(M4)(M3)} a$

b) Since $1/2 > 0$, by Theorem 2.1.7 (c),

$$\frac{1}{2}a < \frac{1}{2}b$$

Again, use Theorem 2.1.7 (b), we have

$$\frac{1}{2}a + \frac{1}{2}a < \frac{1}{2}b + \frac{1}{2}a \quad \text{and} \quad \frac{1}{2}a + \frac{1}{2}b < \frac{1}{2}b + \frac{1}{2}b$$

Using (D) algebraic properties of $\mathbb{R}$, and $(1/2) + (1/2) = 1 \cdot (1/2) + 1 \cdot (1/2) = 2 \cdot (1/2) = 1$, we have

$$a < \frac{1}{2}(a + b) \quad \text{and} \quad \frac{1}{2}(a + b) < b$$

This is exactly what we want.

2.1-23

Clearly the conclusion holds for $n = 1$. So by Mathematical induction, we only need to show when the conclusion holds for $n = k$, then it will hold for $n = k+1$.

First, if we have $a < b$, then by the assumption of induction, we will have $a^k < b^k$(conclusion holds for k). Then

$$a^{k+1} < ab^k < b \cdot b^k = b^{k+1}$$

On the other hand, if we have $a^{k+1} < b^{k+1}$, by the Order Properties, we have three cases, $a > b, a = b, a < b$. we need to rule out first two cases.

Indeed, if $a = b$, we will have $a^{k+1} = b^{k+1}$, which contradicts our assumtion. (You can also show this result by induction). And if $a > b$, then by our conclusion holds for k, we will have $a^k > b^k$ and similar as above, we can get $a^{k+1} > b^{k+1}$, which also contradicts our assumtion. Hence we can only have $a < b$. This will finish our proof.