MATH 2050C Mathematical Analysis I
2020-21 Term 2

Midterm Solution

Question 1

We will show

$$|x_0 + x_1 + \cdots + x_n| \geq |x_0| - (|x_1| + \cdots + |x_n|).$$

by Mathematical induction.

For $n = 1$, using Triangle Inequality ,we have $|x_0 + x_1| \geq |x_0| - |x_1|$. So the above inequality holds for $n = 1$.

Suppose the above inequality holds for $n = k$ with $k \in \mathbb{N}$. That is, we have assumed $|x_0 + x_1 + \cdots + x_k| \geq |x_0| - (|x_1| + \cdots + |x_k|)$. Again, by triangle inequality, we have

$$\begin{aligned}
& |x_0 + x_1 + \cdots + x_k + x_{k+1}| \\
\geq\ & |x_0 + x_1 + \cdots + x_k| - |x_{k+1}| \quad \text{(Triangle inequality)} \\
\geq\ & |x_0| - (|x_1| + \cdots + |x_k|) - |x_{k+1}| \quad \text{(Induction hypothesis)} \\
=\ & |x_0| - (|x_1| + \cdots + |x_{k+1}|)
\end{aligned}$$

Hence the inequality hold for $n = k + 1$. By Mathematical Induction, we know the inequality holds for all $n \in \mathbb{N}$.

Question 2

(a) First, let's show $S \neq \emptyset$. If $x \geq 0$, then clearly $0 \in S$. If $x < 0$, by The Archimedean Property, we know there exists $n_{-x} \in \mathbb{N}$ such that $-x \leq n_x$. So $-n_{-x} \in S$. Hence, S is not empty.

By definition of S, we know x is an upper bound of S since for any $s \in S$, we have $s \leq x$. By the completeness Property of $\mathbb{R}^n$, we know S has a supremum $\sup S$ in $\mathbb{R}^n$.

(b) Let $a = \sup S$. We will show $a \in S$ by contradiction. Suppose $a \notin S$, let's first choose $\varepsilon = \frac{1}{2}$ and since $a - \varepsilon = a - \frac{1}{2}$ is not a upper bound of S by property of supremum, we can find some element $k \in S$ such that $k > a - \frac{1}{2}$. Again, since $a \notin S$, but a is a supremum of S, we have $a > k$. Now choose $\varepsilon' = \frac{a-k}{2}$, we can find another $k' \in S$ such that $k' > a - \varepsilon' = \frac{a+k}{2} \geq k$ since $a - \varepsilon'$ is not a upper bound of S.

Collect above inequality, we have

$$0 < k' - k \leq a - k < a - \left(a - \frac{1}{2}\right) = \frac{1}{2}$$

This is absurd since the distance between any two different integers is at least 1. Hence, we should have $a \in S$. So $\sup S = a$ is an integer.

Question 3

(a) By Density Theorem, we can find the first rational number $r_1 \in \mathbb{Q}$ such that $a < r_1 < b$. Again, we can find the second rational number $r_2 \in \mathbb{Q}$ such that $r_1 < r_2 < b$ still by Density Theorem. One can conductively choose $r_3, r_4, \cdots$ in this way. Or just use $r_{n+2} = \frac{r_{n+1}+r_1}{2} \in \mathbb{Q}$ to define a sequence of rational numbers. Clearly one can check $r_1 < r_n < r_{n-1} < \cdots < r_2$ for any $n \in \mathbb{N}$. So (a, b) contains infinitely many rational numbers.

(b) We will use a variant of Cantor's proof of the uncountability of $\mathbb{R}$ to prove the irrational numbers in (a, b) is uncountable.

Let $r_1 < r_2$ be two rational numbers in (a, b). Consider the map $f(x) = \frac{x-r_1}{r_2-r_1}$. This is a bijective map from (a, b) to $(\frac{a-r_1}{r_2-r_1}, \frac{b-ar_1}{r_2-r_1})$ and it maps rational numbers to rational numbers and maps irrational numbers to irrational numbers. Note that $[0, 1] \subset (\frac{a-r_1}{r_2-r_1}, \frac{b-r_1}{r_2-r_1})$, we only need to show $[0, 1]$ contains uncountable may irrational numbers.

Suppose on contrary, the irrational numbers in $[0, 1]$ is countable. Let $x_1, x_2, \cdots$ be an enumeration of all irrational numbers in $[0, 1]$. Suppose we have the decimal representation for x_n as $x_n = 0.b_{n1}b_{n2} \cdots b_{nn} \cdots$.

Now we want to define a real number $y := 0.y_1y_2 \cdots y_n \cdots$ which is not a rational number. To be precisely, we want to avoid to create a periodic decimal. So we choose y_n in the following ways

$$y_n := \begin{cases} 1, & \text{if } b_{nn} \geq 5 \text{ and } 10^{2k} \leq n < 10^{2k+1}, \\ 3, & \text{if } b_{nn} \geq 5 \text{ and } 10^{2k+1} \leq n < 10^{2k+2}, \\ 6, & \text{if } b_{nn} \leq 4 \text{ and } 10^{2k} \leq n < 10^{2k+1}, \\ 8, & \text{if } b_{nn} \leq 4 \text{ and } 10^{2k+1} \leq n < 10^{2k+2}. \end{cases}$$

Basically, y_n will only take 1 and 6 when $1 \leq n < 10, 100 \leq n < 1000, 10^4 \leq n < 10^5, \cdots$ and y_n will only take 3 and 8 when $10 \leq n < 100, 1000 \leq n < 10000, 10^5 \leq n < 10^6, \cdots$. Clearly, y_n is not periodic. Hence y is not rational and moreover, $y \neq x_n$ for any $n \in \mathbb{N}$. So y is not in the enumeration of $(x_1, x_2, \cdots)$. This contradiction shows $[0, 1]$ and hence (a, b) contains uncountably many irrational numbers.

Question 4

Note that $\left|\frac{n-1}{n}-1\right| = \frac{1}{n}$ and $\left|\frac{n^2+1}{n(n+1)}-1\right| = \frac{n-1}{n(n+1)} = \frac{1}{n} \times \frac{n-1}{n+1} \leq \frac{1}{n}$, we have $|x_n - 1| \leq \frac{1}{n}$ no matter what n is odd or even. Hence for any $\varepsilon > 0$, we can choose $K \in \mathbb{N}$ large enough to make sure $\frac{1}{K} < \varepsilon$ by Archimedean Property. Hence for any $n \geq K$, we have $|x_n - 1| \leq \frac{1}{n} \leq \frac{1}{K} < \varepsilon$. So $\lim(x_n) = 1$.

Question 5

Define $(a_n), (b_n)$ by the following

$$a_n = (-1)^n \frac{1}{n}, \quad b_n = (-1)^n$$

Clearly $(a_n b_n) = (\frac{1}{n})$ will converge to 0. By Squeeze Theorem and $-\frac{1}{n} \leq a_n \leq \frac{1}{n}$, we know $\lim(a_n) = 0$.

For (b_n), we know it has two convergent subsequences $(b_{2k}) = (1)$ and $(b_{2k+1}) = (-1)$ whose limits are not equal. Hence (b_n) is divergent.

Question 6

(a) By the well known formula $\sum_{k=1}^{n} k^2 = \frac{n(2n+1)(n+1)}{6}$, we have

$$x_n = \frac{n(2n+1)(n+1)}{n^3} = \frac{2n^2+3n+1}{6n^2}.$$

Hence

$$\left|x_n - \frac{1}{3}\right| = \frac{3n+1}{6n^2} < \frac{3n+3n}{6n^2} = \frac{1}{n}.$$

So for any $\varepsilon > 0$, we can choose $K \in \mathbb{N}$ such that $\frac{1}{K} < \varepsilon$ by Archimedean Property. So we have

$$\left|x_n - \frac{1}{3}\right| \leq \frac{1}{K} < \varepsilon$$

for any $n \geq K$ with $n \in \mathbb{N}$. Hence $\lim(x_n) = \frac{1}{3}$.

(b) We note

$$\begin{aligned} y_n &= \sum_{k=1}^{n} \frac{(2k-1)^2}{n^3} = \sum_{k=1}^{n} \frac{4k^2-4k+1}{n^3} = 4x_n - 4 \times \frac{n(n+1)}{2n^3} + \frac{n}{n^3} \\ &= 4x_n - \frac{2n+1}{n^2} \end{aligned}$$

Suppose $z_n = \frac{2n+1}{n^2}$. Note $0 \leq z_n \leq \frac{3}{n}$, so by the Squeeze Theorem, we know $0 \leq \lim(z_n) \leq \lim \frac{3}{n} = 0$. Hence $\lim(y_n) = 4\lim(x_n) - \lim(z_n) = \frac{4}{3}$.

Question 7

(a) We prove the following claim by Mathematical Induction.
Claim. $x_n < 1$ and $x_n < x_{n+1}$ for all $n \in \mathbb{N}$.

First, the claim holds for $n = 1$ since $x_1 = a = \frac{1}{2} < 1$ and $x_2 = \frac{1}{7}(x_1^3 + 6) < \frac{1}{7}(1^3 + 6) = 7$.

Suppose the claim holds for $n = k$ with $k \in \mathbb{N}$. That is, we assume $x_k < 1$ and $x_k < x_{k+1}$. So we will have

$$x_{k+1} = \frac{1}{7}\left(x_k^3 + 6\right) < \frac{1}{7}\left(1^3 + 6\right) = 1$$

and

$$x_{k+1} = \frac{1}{7}\left(x_k^3 + 6\right) < \frac{1}{7}\left(x_{k+1}^3 + 6\right)\big) = x_{k+2}.$$

Hence the claim holds for $n = k + 1$. By Mathematica Induction, we know the claim holds for all $n \in \mathbb{N}$.

So we get (x_n) is an increasing sequence and bounded above by 1. Hence by Monotone Convergence Theorem, we know $\lim(x_n)$ exists. Suppose $x = \lim(x_n)$

Let we take limit at both side of equation $x_n = \frac{1}{7}(x_n^3 + 6)$, we will have $x = \frac{1}{7}(x^3 + 6)$. This equality has three solutions $-3, 1, 2$. Since $\frac{1}{2} \leq x_n < 1$, x cannot be $-3, 2$. Hence $\lim x_n = 1$.

(b) When $a = 7$, (x_n) will be divergent. First, we know (x_n) is increasing again by Mathematical Induction. Moreover, we have $x_{n+1} = \frac{1}{7}(x_n^3 + 6) > \frac{1}{7} \times 7^2 x_n = 7x_n$. Hence $x_n \geq 7^n$. So (x_n) is not a bounded sequence. So (x_n) is divergent.

Question 8

(a) First, let's show

$$\limsup(x_n + y_n) \geq \lim(x_n) + \limsup(y_n).$$

Let S be the set of subsequential limits of $(x_n + y_n)$. So we know $\sup S = \limsup(x_n + y_n)$. Similarly, let S' be the set of subsequential limit of (y_n) and let $x = \lim(x_n)$.

Now for any $s' \in S'$, we can find a subsequence (y_{n_k}) such that $\lim(y_{n_k}) = s'$. Note $\lim(x_{n_k}) = x$ since (x_n) is a convergence sequence, we know $(x_{n_k} + y_{n_k})$ is convergent with $\lim(x_{n_k} + y_{n_k}) = s' + x$. So $s' + x \in S$. This implies $s' + x \leq \sup S$ and hence $s' \leq \sup S - x$. By the definition of supremum, we know $\sup S' \leq \sup S - x$. This is exactly $\limsup(x_n + y_n) \geq \lim(x_n) + \limsup(y_n)$.

For the inequality of another direction, we just note

$$\begin{aligned}\limsup(y_n) = & \limsup(y_n + x_n - x_n) \geq \limsup(y_n + x_n) + \lim(-x_n) \\ = & \limsup(y_n + x_n) - \lim(x_n)\end{aligned}$$

where we've used $\lim(-x_n)$ exists.

Combining above, we have

$$\limsup(x_n + y_n) = \lim(x_n) + \limsup(y_n).$$

(b) We choose $x_n = (-1)^n$, $y_n = (-1)^{n+1}$. So we have

$$\limsup(x_n) = \limsup(y_n) = 1$$

but

$$\limsup(x_n + y_n) = \limsup(0) = 0$$

Hence

$$\limsup(x_n + y_n) < \limsup(x_n) + \limsup(y_n).$$