MATH 2050C Lecture 5 (Jan 26)

Last week: Completeness of $\mathbb{R}$, existence of infimum.

some conseq. of this property.

§ Consequences of Completeness Property (Textbook §2.4)

Recall: "Archimedean Property".

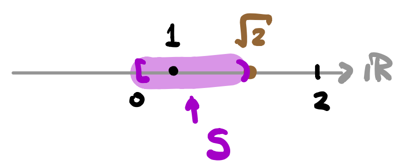
- $\mathbb{N} \subseteq \mathbb{R}$ is NOT bdd above
- $\forall t > 0$, $\exists n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < t$
- $\forall y > 0$, $\exists n \in \mathbb{N}$ st. $n - 1 \leq y < n$

Recall: $\sqrt{2} \notin \mathbb{Q} \subsetneq \mathbb{R}$

Thm: (Existence of $\sqrt{2}$ in $\mathbb{R}$)

$\exists x \in \mathbb{R}$ st. $x > 0$ and $x^2 = 2$.

Picture:



Proof: Let $S := \{ s \in \mathbb{R} : s \geq 0, s^2 < 2 \}$

Claim 1: $S \neq \phi$ $(\because 0 \in S)$

Claim 2: S is bdd above.

Why? $\forall s \in S$, $s \geq 0$ and "$s^2 < 2 < 4 = 2^2 \overset{s \geq 0}{\Rightarrow} s < 2$"

i.e. 2 is an upper bd for S

By Completeness Property, $x := \sup S \in \mathbb{R}$ exists.

Claim 3: $x > 0$ and $\boxed{x^2 = 2.}$

Since $1 \in S$, and x is an upper bd for S.

$0 < 1 \leq x$ Thus, $x > 0$.

To prove $x^2 = 2$, we argue by contradiction.

Suppose NOT, by Trichotomy, either $x^2 < 2$ OR $x^2 > 2$.

Case 1: $x^2 < 2$

WANT: Find $n \in \mathbb{N}$ s.t. $x + \frac{1}{n} \in S$ ($\Rightarrow x$ is NOT an upper bd for S) contradicting $x = \sup S$

i.e. $\left(x + \frac{1}{n}\right)^2 < 2$.

$$\left(x+\frac{1}{n}\right)^2 < 2$$
$$\Uparrow$$
$$x^2 + \frac{2x}{n} + \frac{1}{n^2} < 2$$
$$\Uparrow$$
$$x^2 + \frac{2x}{n} + \frac{1}{n} < 2$$
$$\Uparrow$$
$$\frac{2x+1}{n} < 2 - x^2$$
$$\Uparrow$$
$$\frac{1}{n} < \frac{2-x^2}{2x+1}$$

Proof?

Now, by assumption $2 - x^2 > 0$.

also $x > 0 \Rightarrow 2x + 1 > 0$

Thus, $\frac{2-x^2}{2x+1} > 0$.

By Archimedean Property, $\exists\, n \in \mathbb{N}$ st.

$$0 < \frac{1}{n} < \frac{2-x^2}{2x+1} \quad \cdots\cdots (*)$$

Then, for this n,

$$\left(x+\frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2}$$

$$\left(\because \frac{1}{n^2} \leq \frac{1}{n} \;\; \forall n \in \mathbb{N},\ n \geq 1\right) \leq x^2 + \frac{2x}{n} + \frac{1}{n}$$

$$= x^2 + \frac{2x+1}{n} \underset{\text{by } (*)}{<} 2$$

Case 2: $x^2 > 2$.

Want: Find $m \in \mathbb{N}$ s.t. $x - \frac{1}{m}$ is an upper bd for S

($\Rightarrow x$ is NOT the least upper bd. contradicting $x = \sup S$)

Arch. Property: Choose $m \in \mathbb{N}$ s.t. $\frac{1}{m} < \frac{x^2-2}{2x}$ (R.H.S. > 0)

$(\because \frac{1}{m^2} > 0)$ $\qquad \forall s \in S$

$$\left(x - \frac{1}{m}\right)^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2} > x^2 - \frac{2x}{m} \geq 2 > s^2$$

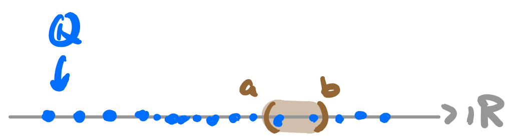
$\blacksquare$

Thm: (Density of $\mathbb{Q}$ in $\mathbb{R}$)

For any $a, b \in \mathbb{R}$ st. $a < b$.

$\exists\, x \in \mathbb{Q}$ st. $a < x < b$.

Picture



Proof: Given $a, b \in \mathbb{R}$, $0 < a < b$, then $b - a > 0$.

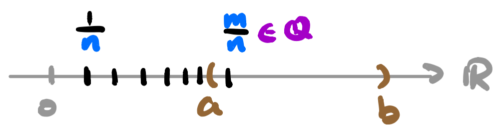
Step size

By Archimedean Property, $\exists\, n \in \mathbb{N}$ st. $0 < \frac{1}{n} < b - a$

Since $na > 0$, by Archimedean Property,

$\exists\, m \in \mathbb{N}$ st. $m - 1 \le na < m$.

Picture:



Note: $\frac{1}{n} < b - a \Rightarrow na + 1 < nb$

$m - 1 \le na < m \Rightarrow m \le na + 1 < m + 1$

Combining these two inequalities,

$$na < m \le na + 1 < nb$$

Divide by $n \Rightarrow a < \underbrace{\frac{m}{n}}_{\in \mathbb{Q}} < b$.

$\square$

Cor: $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$

Pf: Fix any $a, b \in \mathbb{R}$, want: $\exists\, y \in \mathbb{R} \setminus \mathbb{Q}$ st. $a < y < b$.
$(a < b)$.

Consider $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$ in $\mathbb{R}$, by density of $\mathbb{Q}$ in $\mathbb{R}$,

$\exists\, q \in \mathbb{Q}$ st. $\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}$

$$\Rightarrow a < \underbrace{q \cdot \sqrt{2}}_{\notin \mathbb{Q}} < b$$

$\square$

§ Intervals (Textbook § 2.5)

$\exists$ 9 types of intervals (closed / open, bdd / unbdd)

Given $a, b \in \mathbb{R}$, $a < b$.

Notation:

$(a,b) := \{x \in \mathbb{R} \mid a < x < b\}$

$[a,b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$

$(a,b] := \{x \in \mathbb{R} \mid a < x \leq b\}$

$[a,b) := \{x \in \mathbb{R} \mid a \leq x < b\}$.

"bdd intervals"

$(a,\infty) := \{x \in \mathbb{R} \mid a < x\}$

$[a,\infty) := \{x \in \mathbb{R} \mid a \leq x\}$.

$(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}$

$(-\infty, b] := \{x \in \mathbb{R} \mid x \leq b\}$

$(-\infty, \infty) =: \mathbb{R}$

"unbdd intervals"

Def$^{\underline{n}}$: Length $(I) := b - a > 0$.

Q: When is $S \subseteq \mathbb{R}$ an "interval"?

A: "connectedness" (MATH 3070)

Thm: (Characterization of intervals)

Let $S \subseteq \mathbb{R}$. Suppose

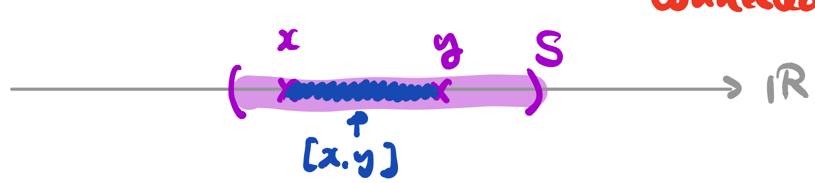
(i) $\exists\, s_1, s_2 \in S$ s.t. $s_1 \neq s_2$

"Connected" *(ii) If $x, y \in S$, $x < y$, then $[x,y] \subseteq S$.

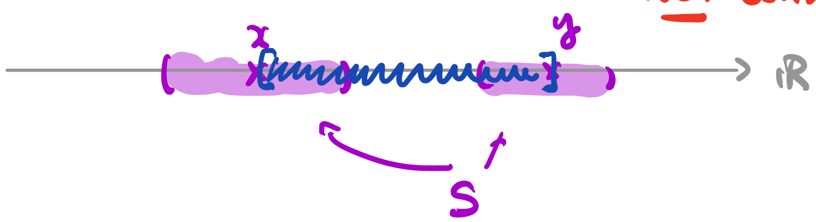
Then, S is an interval. [Note: could be unbdd.]

Picture:

Connected!

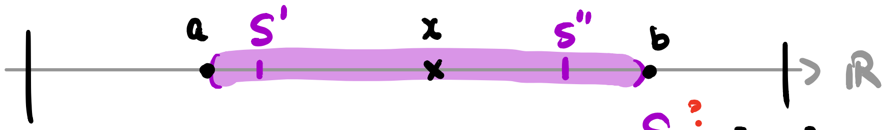


NOT connected!



Proof: We just treat the case $S \subseteq \mathbb{R}$ is bdd.

Picture:



$S \overset{?}{=} [a,b]$ or (a,b) or $[a,b)$ or $(a,b]$

Completeness Property $\Rightarrow$ $a := \inf S$, $b := \sup S$ exist in $\mathbb{R}$

By (i), we have $a \leq s_1 < s_2 \leq b \Rightarrow a < b$.

Claim: $(a,b) \subseteq S$

Pf of Claim: Take any $x \in (a,b)$, ie $a < x < b$

Want to show: $x \in S$.

Since $x > a = \inf S$, it cannot be a lower bd of S.

i.e. $\exists\, s' \in S$ s.t. $s' < x$

Since $x < b = \sup S$, it cannot be an upper bd of S

i.e. $\exists\, s'' \in S$ s.t. $x < s''$

By (ii), $[s', s''] \subseteq S$ but $x \in [s', s''] \Rightarrow x \in S$.

This implies $S = (a,b)$ or $[a,b)$ or $(a,b]$ or $[a,b]$.

depending on whether $\inf S = a \in S$ or $\sup S = b \in S$.

$\blacksquare$