THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 7 (October 28, 30)

1 One-Sided Limits

Definition. Let $A \subseteq \mathbb{R}$, $f\colon A \to \mathbb{R}$.

(i) If c is a cluster point of $A \cap (c, \infty)$, then we say that $L \in \mathbb{R}$ is a **right-hand limit of f at c** and write
$$\lim_{x \to c^+} f(x) = L$$
if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in A$ with $0 < x - c < \delta$, then $|f(x) - L| < \varepsilon$.

(ii) If c is a cluster point of $A \cap (-\infty, c)$, then we say that $L \in \mathbb{R}$ is a **left-hand limit of f at c** and write
$$\lim_{x \to c^-} f(x) = L$$
if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in A$ with $0 < c - x < \delta$, then $|f(x) - L| < \varepsilon$.

Theorem 1. *Let $A \subseteq \mathbb{R}$, $f\colon A \to \mathbb{R}$. Suppose c is a cluster point of both $A \cap (c, \infty)$ and $A \cap (-\infty, c)$. Then $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^+} f(x) = L = \lim_{x \to c^-} f(x)$*

Example 1. Let $g(x) = e^{1/x}$ for $x \neq 0$. Consider the limits $\lim_{x \to 0^+} g(x)$ and $\lim_{x \to 0^-} g(x)$.

2 Infinite Limits

Definition. Let $A \subseteq \mathbb{R}$, $f\colon A \to \mathbb{R}$, and $c \in \mathbb{R}$ be a cluster point of A.

(i) We say that f **tends to ∞ as** $x \to c$, and write
$$\lim_{x \to c} f(x) = \infty,$$
if for every $\alpha \in \mathbb{R}$, there exists $\delta > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) > \alpha$.

(ii) We say that f **tends to $-\infty$ as** $x \to c$, and write
$$\lim_{x \to c} f(x) = -\infty,$$
if for every $\beta \in \mathbb{R}$, there exists $\delta > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) < \beta$.

Remarks. The following are defined in a similar fashion:

$$\lim_{x\to c^+} f = \infty, \quad \lim_{x\to c^-} f = \infty, \quad \lim_{x\to c^+} f = -\infty, \quad \lim_{x\to c^-} f = -\infty.$$

Example 2. Evaluate the limits $\lim_{x\to 1^-} \dfrac{x}{\sqrt{x}-x}$ and $\lim_{x\to 1^+} \dfrac{x}{\sqrt{x}-x}$ using definition. What can you say about the limit $\lim_{x\to 1} \dfrac{x}{\sqrt{x}-x}$?

Example 3. Is there a function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to c} f(x) = \infty$ for every $c \in \mathbb{R}$.

Solution: No. Suppose there is such a function f. Then, given any $c \in \mathbb{R}$ and $M > 0$, there exists $\delta > 0$ such that $f(x) \geq M$ whenever $x \in V_\delta(c) \setminus \{x\}$. By shrinking the neighbourhood if necessary, we can easily deduce the following:

Claim: Suppose $a < b$. For any $M > 0$, there are α, β with $a < \alpha < \beta < b$ such that $$f(x) \geq M \qquad \text{whenever } x \in [\alpha, \beta].$$

Let $I_0 = [0, 1]$. By the claim, there are $0 < \alpha_1 < \beta_1 < 1$ such that

$$f(x) \geq 1 \qquad \text{whenever } x \in [\alpha_1, \beta_1].$$

Let $I_1 := [\alpha_1, \beta_1]$. By the claim again, there are $\alpha_1 < \alpha_2 < \beta_2 < \beta_1$ such that

$$f(x) \geq 2 \qquad \text{whenever } x \in [\alpha_2, \beta_2].$$

Continue in this way, we can find a sequence $\{I_n\}_{n\in\mathbb{N}}$ of closed bounded intervals such that

(i) $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$, and

(ii) $f(x) \geq n$ for any $x \in I_n$.

By the Nested Interval Theorem, $\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$. Let $x_0 \in \bigcap_{n\in\mathbb{N}} I_n$. Then we have $f(x_0) \geq n$ for all $n \in \mathbb{N}$, contradicting the fact that $f(x_0) \in \mathbb{R}$.

3 Limits at Infinity

Definition. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Suppose that $(a, \infty) \subset A$ for some $a \in \mathbb{R}$. We say that $L \in \mathbb{R}$ is a **limiit of f as** $x \to \infty$, and write

$$\lim_{x\to\infty} f(x) = L,$$

if given any $\varepsilon > 0$ there exists $K = K(\varepsilon) > a$ such that for any $x > K$, then $|f(x) - L| < \varepsilon$.

Remarks. $\lim_{x\to-\infty} f(x) = L$ is defined similarly.

Example 4. By virtue of definition, show that $\lim_{x\to\infty} \dfrac{\sqrt{x}-x}{\sqrt{x}+x} = -1$.