THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 6 (October 21, 23)

1 Cluster Points

Definition. Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is said to be a **cluster point** of A if given any $\delta > 0$, there exists $x \in A$, $x \neq c$ such that $|x - c| < \delta$.

Remarks. 1. A cluster point may or may not be an element of A.

2. Equivalently, c is a cluster point of A if and only if $V_\delta(c) \cap A \setminus \{c\} \neq \emptyset$ for any $\delta > 0$.

3. We denote the set of cluster points of A by A^c.

Example 1. Find the set of cluster points of the following subsets of $\mathbb{R}$.

(a) $A = \mathbb{N}$

(b) $B = \{\frac{1}{n} : n \in \mathbb{N}\}$

Solution. (a) Let $c \in \mathbb{R}$. If $c \in A$, then $V_\delta(c) \cap A \setminus \{c\} = \emptyset$ for $\delta = 1/2$. If $c \notin A$, then $V_\delta(c) \cap A \setminus \{c\} = \emptyset$ for $\delta = \min\{c - \lfloor c \rfloor, \lfloor c \rfloor + 1 - c\} > 0$. So $c \notin A^c$. Hence $A^c = \emptyset$.

(b) By similar arguments, we can show that $B^c = \{0\}$.

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2 Limits of Functions

Definition. Let $A \subseteq \mathbb{R}$, and let c be a cluster point of A. For a function $f : A \to \mathbb{R}$, a real number L is said to be a **limit of f at c** if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

In this case, the limit is in fact unique and we write

$$\lim_{x \to c} f(x) = L \quad \text{or} \quad \lim_{A \ni x \to c} f(x) = L.$$

Example 2. By virtue of ε-δ definition, show that $\displaystyle\lim_{x \to 2} \frac{x+6}{x^2-2} = 4$.

Solution. Clearly $f(x) := \dfrac{x+6}{x^2-2}$ has a natural domain $\mathbb{R} \setminus \{\pm\sqrt{2}\}$, which has 2 as a cluster point.

For $x \in \mathbb{R} \setminus \{\pm\sqrt{2}\}$,

$$|f(x) - 4| = \left| \frac{x+6}{x^2-2} - 4 \right| = \frac{|4x^2 - x - 14|}{|x^2 - 2|} = \frac{|4x+7|}{|x^2-2|} \cdot |x - 2|.$$

If $|x-2| < \dfrac{1}{2}$, then

$$\frac{3}{2} < x < \frac{5}{2} \implies \frac{1}{4} < x^2 - 2 < \frac{17}{4},$$

and

$$|4x+7| = |4(x-2)+15| \leq 4|x-2| + 15 \leq 20.$$

Let $\varepsilon > 0$ be given. Take $\delta := \min\left\{\dfrac{\varepsilon}{80}, \dfrac{1}{2}\right\}$. Now if $0 < |x-2| < \delta$, then

$$|f(x) - 4| = \frac{|4x+7|}{|x^2-2|} \cdot |x-2| < 80 \cdot \frac{\varepsilon}{80} = \varepsilon.$$

Hence $\lim_{x\to 2} f(x) = 4$. ◀

Theorem 1. *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $c \in A^c$ and $L \in \mathbb{R}$.*

(1) (Sequential Criterion) $\lim_{x\to c} f(x) = L$ if and only if

$$\lim_n f(x_n) = L \textit{ whenever } (x_n) \textit{ is a sequence in } A \setminus \{c\} \textit{ convergent to } c.$$

(2) (Divergence Criterion) f does not have a limit at c if and only if

$$\textit{there exists a sequence } (x_n) \textit{ in } A \setminus \{c\} \textit{ convergent to } c \textit{ but } (f(x_n)) \textit{ does not converge in } \mathbb{R}.$$

Example 3. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Assume that $\lim_{x\to 0} f(x) = L$ exists. Prove that $L = 0$, and then prove that f has a limit at every point $c \in \mathbb{R}$.

Example 4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that $\lim_{x\to c} f(x)$ does not exist for every $c \in \mathbb{R}$.

Solution. Let $c \in \mathbb{R}$. In view of the Divergence Criterion, we want to construct a sequence (x_n) in $\mathbb{R} \setminus \{c\}$ such that $x_n \to c$ but $(f(x_n))$ diverges.

By the density of $\mathbb{Q}$, for each $n \in \mathbb{N}$, there exists $r_n \in \mathbb{Q}$ such that

$$c < r_n < c + \frac{1}{n}.$$

Similarly, by the density of $\mathbb{R} \setminus \mathbb{Q}$, for each $n \in \mathbb{N}$, there exists $s_n \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$c < s_n < c + \frac{1}{n}.$$

Now let (x_n) be the sequence given by $x_n = \begin{cases} r_n & \text{if } n \text{ is odd} \\ s_n & \text{if } n \text{ is even.} \end{cases}$

Then clearly, $\lim(x_n) = c$ and $x_n \neq c$ for all $n \in \mathbb{N}$.

However,

$$f(x_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

so that $\lim(f(x_n))$ does not exist. By Divergence Criterion, $\lim\limits_{x \to c} f(x)$ does not exist. ◀