THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 4 (October 7, 9)

1 Subsequences and the Bolzano-Weierstrass Theorem

Definition. Let (x_n) be a sequence of real numbers and let $n_1 < n_2 < \cdots < n_k < \cdots$ be a **strictly increasing** sequence of natural numbers. Then the sequence (x_{n_k}) is called a **subsequence** of (x_n).

Subsequence Theorem. *If (x_n) converges, then any subsequence (x_{n_k}) of (x_n) also converges to the same limit.*

Theorem 1. *Let (x_n) be a sequence of real numbers. Then the following are equivalent:*

(i) (x_n) *does not converge to* $x \in \mathbb{R}$.

(ii) There exists $\varepsilon_0 > 0$ *such that for any* $k \in \mathbb{N}$, *there exists* $n_k \in \mathbb{N}$ *such that* $n_k \geq k$ *and* $|x_{n_k} - x| \geq \varepsilon_0$.

(iii) There exists $\varepsilon_0 > 0$ *and a subsequence* (x_{n_k}) *of* (x_n) *such that* $|x_{n_k} - x| \geq \varepsilon_0$ *for all* $k \in \mathbb{N}$.

Example 1. Let $\ell \in \mathbb{R}$. Show that a sequence (x_n) converges to ℓ if and only if every subsequence of (x_n) has a further subsequence that converges to ℓ

Example 2. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$.

Solution. As (x_n) is unbounded, we have $\forall M > 0$, $\exists\, n \in \mathbb{N}$ such that $|x_n| > M$.

Pick $n_1 \in \mathbb{N}$ such that $|x_{n_1}| > 1$.
Then pick $n_2 \in \mathbb{N}$ such that $|x_{n_2}| > \max\{2, |x_1|, |x_2|, \ldots, |x_{n_1}|\}$. So $|1/x_{n_2}| < 1/2$ and $n_2 > n_1$.

Suppose $n_1 < n_2 < \cdots < n_k$ are chosen so that $|1/x_{n_j}| < 1/j$ for $1 \leq j \leq k$.

Pick $n_{k+1} \in \mathbb{N}$ such that $|x_{n_{k+1}}| > \max\{k+1, |x_1|, |x_2|, \ldots, |x_{n_k}|\}$. So $|1/x_{n_{k+1}}| < 1/(k+1)$ and $n_{k+1} > n_k$.
Continue in this way, we obtain a subsequence (x_{n_k}) of (x_n) such that

$$|1/x_{n_k}| < 1/k \qquad \text{for all } k \in \mathbb{N}.$$

Now $\lim(1/x_{n_k}) = 0$ follows immediately from Squeeze Theorem.

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The Bolzano-Weierstrass Theorem. *A bounded sequence of real numbers has a convergent subsequence*

Example 3. Prove that a bounded divergent sequence has two subsequences converging to different limits.

Solution. Let (x_n) be a bounded divergent sequence. In particular, any subsequence of (x_n) is also bounded. By Bolzano-Weierstrass Theorem, (x_n) has a convergent subsequence (x_{n_k}). Suppose $\lim(x_{n_k}) = \ell$. Since (x_n) does not converge to ℓ, there are $\varepsilon_0 > 0$ and another subsequence (x_{m_k}) of (x_n) such that

$$|x_{m_k} - \ell| \geq \varepsilon_0 \quad \text{for all } k. \tag{\#}$$

By Bolzano-Weierstrass Theorem again, (x_{m_k}) has a further subsequence $(x_{m_{k_j}})$ that converges to some real number ℓ'. By (#), $\ell \neq \ell'$. Now (x_{n_k}) and $(x_{m_{k_j}})$ are the desired subsequences of (x_n). ◀

2 Limit Superior and Limit Inferior

Let (a_n) be a bounded sequence of real numbers. For each $n \in \mathbb{N}$, define

$$t_n = \sup_{m \geq n} a_m = \sup\{a_m : m \geq n\} \quad \text{and} \quad s_n = \inf_{m \geq n} a_m = \inf\{a_m : m \geq n\}.$$

Then, as required in HW, one can show that (t_n) and (s_n) are both monotone and convergent.

Definition. The **limit superior** and **limit inferior** of (a_n) are defined, respectively, by

$$\limsup_n a_n := \lim_n t_n = \inf_{n \geq 1} \left(\sup_{m \geq n} a_m \right),$$

$$\liminf_n a_n := \lim_n s_n = \sup_{n \geq 1} \left(\inf_{m \geq n} a_m \right).$$

Proposition 2. *Let (a_n) be a bounded sequence of real numbers. Then*

(a) $\liminf_n a_n \leq \limsup_n a_n$.

(b) (a_n) converges to ℓ if and only if $\limsup_n a_n = \liminf_n a_n = \ell$.

Example 4. Let (x_n) and (y_n) be bounded sequences of real numbers. Show that

(a) $\limsup_n (-x_n) = -\liminf_n x_n$;

(b) if $x_n \leq y_n$ for all n, then $\limsup_n x_n \leq \limsup_n y_n$ and $\liminf_n x_n \leq \liminf_n y_n$;

(c) $\liminf_n x_n + \liminf_n y_n \leq \liminf_n (x_n + y_n) \leq \limsup_n (x_n + y_n) \leq \limsup_n x_n + \limsup_n y_n$.

Example 5. Let (x_n) be a sequence of positive real numbers. Show that

$$\liminf_n \frac{x_{n+1}}{x_n} \leq \liminf_n \sqrt[n]{x_n} \leq \limsup_n \sqrt[n]{x_n} \leq \limsup_n \frac{x_{n+1}}{x_n}.$$

Solution. We only prove the last inequality. Assume $\limsup_n \frac{x_{n+1}}{x_n} < +\infty$.

Let $\alpha > \limsup_n \frac{x_{n+1}}{x_n} = \inf_{n\geq 1}\left(\sup_{m\geq n} \frac{x_{m+1}}{x_m}\right)$.

Then there exists $n \in \mathbb{N}$ such that $\frac{x_{m+1}}{x_m} < \alpha$ for all $m \geq n$. Hence, for $m \geq n+1$,

$$\frac{x_m}{x_n} = \frac{x_{n+1}}{x_n} \cdot \frac{x_{n+2}}{x_{n+1}} \cdots \frac{x_m}{x_{m-1}} < \alpha^{m-n},$$

so that

$$\sqrt[m]{x_m} < \alpha^{1-\frac{n}{m}} x_n^{\frac{1}{m}} = \alpha \cdot \sqrt[m]{C},$$

where $C = x_n \alpha^{-n}$. Now

$$\limsup_m \sqrt[m]{x_m} \leq \limsup_m (\alpha \cdot \sqrt[m]{C}) = \lim_m (\alpha \cdot \sqrt[m]{C}) = \alpha.$$

Since α is arbitrary, we have

$$\limsup_n \sqrt[n]{x_n} \leq \limsup_n \frac{x_{n+1}}{x_n}.$$

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