MATH2050B 2021 HW(Deadline 18 Sept)
TA's solutions[1] to selected problems

Q1. Let $a, b \in \mathbb{R}$ such that

$$a \leq b + 50\epsilon \text{ for all } \epsilon > 0$$

Show that $a \leq b$.

Solution. Suppose on the contrary that $a > b$, i.e. $a - b > 0$. Then there exists $\delta \in \mathbb{R}$ such that $0 < \delta < a - b$. Let $\epsilon = \delta/50 > 0$. Then by the assumption we have

$$a \leq b + 50\epsilon = b + \delta$$

Then $a - b \leq \delta$, which contradicets to the assumption that $\delta < a - b$.

Q2. Let a, b, c be positive real numbers such that $a^2 < b < c^2$. Show that there exists a natural number $N \geq 1997$ such that

$$(a + \frac{1}{N})^2 < b < (c - \frac{1}{N})^2$$

Solution. Let $b - a^2 = \epsilon$. We know that $\epsilon > 0$. By Archimedean Property, there exists a natural number M_1 such that $\frac{1}{M_1} < \frac{\epsilon}{4a}$, i.e. $\frac{2a}{M_1} < \frac{\epsilon}{2}$. By Archimedean Property there exists a nautral number M_2 such that $\frac{1}{M} < \frac{\epsilon}{2}$

Let $M = \max(M_1, M_2, 1997)$. Then M satisfies

$$a^2 + \frac{2a}{M} + \frac{1}{M^2} < a^2 + \epsilon = b$$

Similarly there exists a natural number $M' \geq 1997$ such that

$$b < (c - \frac{1}{N})^2$$

Then the desired number is $\max(M, M')$.

Q3. Show that $(m, m+1) \cap \mathbb{Z} = \varnothing$ for all $m \in \mathbb{Z}$.

Solution. Let $P(m)$ be the proposition that $(m, m+1) \cap \mathbb{Z} = \varnothing$.

We claim that $P(0)$ is true. Suppose $P(0)$ is not true, then there exists $a \in \mathbb{Z}$ such that $0 < a < 1$. Let $S = \{k \in \mathbb{Z}_{\geq 0} : 0 < k < 1\}$. Then $S \neq \varnothing$. By the well-ordering principle, $x := \min(S)$ exists. We have $0 < x < 1$, so $0 < x^2 < 1$, but this contradicts to the minimality of x. So $P(0)$ must be true in the first place.

Suppose that $P(m)$ is true for some $m \in \mathbb{Z}_{\geq 0}$, we claim that $P(m+1)$ is true. Suppose that $P(m+1)$ is not true, then there exists $a \in \mathbb{Z}_{\geq 0}$ such that $m+1 < a < m+2$. Then $a-1$ is an integer such that $m < a < m+1$, but this contradicts $P(m)$, so $P(m+1)$ must be true.

Suppose that$P(m)$ is true for some $m \in \mathbb{Z}_{\geq 0}$, similar to the above we can show that $P(m-1)$ is true.

By the Principle of MI, $P(m)$ is true for all $m \in \mathbb{Z}$.

[1]please kindly send an email to `nclliu@math.cuhk.edu.hk` if you have spotted any typo/error/mistake.

Q4. Suppose that X is a non-empty subset of $\mathbb{R}$ and $\alpha := \inf X$ exists in $\mathbb{R}$. Show that $-\alpha = \sup(-X)$.

Solution. We first claim that $-\alpha$ is an upper bound of $-X$. Let $y \in -X$, there is some $x \in X$ such that $y = -x$. Then $\alpha \geq y$ by definition of α, so $-y \leq -\alpha$.

Let $\epsilon > 0$. To show $-\alpha$ is the sup of $-X$ it remains to show there exists $y \in -X$ such that $-\alpha - \epsilon < y$.

Note $\alpha = \inf(X)$, there exists $x \in X$ such that $x < \alpha + \epsilon$, i.e. $-\alpha - \epsilon < -x$. Take $y = -x \in -X$, then we are done.

Q5. Let $\varnothing \neq A, B \subset \mathbb{R}$. Show that

$$\inf(A + B) = \inf A + \inf B$$

provided that (1) $\inf(A + B)$ exists in $\mathbb{R}$ or (2) both $\inf A$ and $\inf B$ exist.

Solution. Case 1. Suppose that $\inf(A+B)$ exists. We first claim that $\inf A$ exists. Let $a \in A$.

Fix an element $b \in B$, then $\inf(A + B) \leq a + b$. This shows that A is bounded below by $\inf(A + B) - b$. Hence $\inf A$ exists, similarly $\inf B$ exists.

Now pick any $a \in A, b \in B$, then

$$\inf(A + B) - b \leq a$$

Note the above holds for all $a \in A$. It follows that

$$\inf_{a \in A} \Big(\inf(A + B) - b \Big) \leq \inf_{a \in A} a$$

Note $\inf(A + B) - b$ is a constant independent of the set A, so LHS is $\inf(A + B) - b$. Thus $\inf(A + B) \leq \inf A + b$. Note that this inequality holds for all $b \in B$, so

$$\inf(A + B) = \inf_{b \in B}(\inf(A + B)) \leq \inf_{b \in B}(\inf A + b) = \inf A + \inf B$$

On the other hand, if $x \in A+B$, then $\inf A + \inf B \leq x$ because $x = a+b$ for some $a \in A$, $b \in B$. So $\inf A + \inf B \leq \inf(A + B)$. Hence $\inf(A + B) = \inf A + \inf B$.

Case 2. Suppose that $\inf A$ and $\inf B$ exists, then $A + B$ is bounded below by $\inf A + \inf B$. Using **Case 1** it follows that

$$\inf(A + B) = \inf A + \inf B$$