MATH2050B 2021 Assignment 1
TA's solutions[1] to selected problems

Q1. (How to find square root for $a > 0$) Let $a > 0$. Since $x > 0$ is a square root of a iff $x = a/x$ or $x = 1/2(x + a/x)$. Pick $x_0 > 0$ and define $x_n = g(x_{n-1})$ for all n, where $g : (0, \infty) \to (0, \infty)$ is defined by $g(t) = 1/2(t + a/t)$. Show that

(i) $g(t)^2 \geq a$ for all $t > 0$

(ii) If $x > 0$ and $x^2 \geq a$ then $x \geq g(x)$

(iii) $lim_n x_n$ exists in $\mathbb{R}$ (and find the limit)

Solution. (i) Let $t > 0$. $g(t)^2 - a = \frac{1}{4}(t^2 + 2a + \frac{a^2}{t^2}) - a = \frac{1}{4}(t^2 - 2a + \frac{a^2}{t^2}) = \frac{1}{4}(t - \frac{a}{t})^2 \geq 0$. For (ii), let $x > 0$ satisfy $x^2 \geq a$. Then

$$x - g(x) = \frac{x}{2} - \frac{a}{2x} = \frac{1}{2}\frac{x^2 - a}{x} \geq 0$$

For (iii), we show that the sequence $(x_n)_{n=1}^{\infty}$ is decreasing. By (i) we know that $x_1^2 \geq a$. Note that $x_1 > 0$, by (ii) we know that $x_1 \geq g(x_1) = x_2$. Note by (i) again $x_2^2 \geq a$, and $x_2 > 0$, by (ii) we know that $x_2 \geq g(x_2) = x_3$. Using the same argument, $(x_n)_{n=1}^{\infty}$ is decreasing. Moreover it is bounded below by 0. By MCT $\lim_n x_n$ exists in $\mathbb{R}$.

To find the limit, first note that

$$x_{n-1} - \frac{x_n}{2} = \frac{a}{2x_n}$$

Because the LHS is convergent, so the RHS is convergent as well. Moreover if $\lim_n x_n = x$ then $x > 0$ and $\frac{x}{2} = \frac{a}{2x}$. Hence x is the square root of a.

Q2. Let $1 < N \in \mathbb{N}$. We assume the existence of positive N-th root $r^{1/N}$ of any positive number r and $r^{1/N} > 0$

(i) Let $x, y > 0$. Show that

$$|x - y| \leq \frac{|x^N - y^N|}{y^{N-1}}$$

(ii) Suppose $a_n > 0$ for all n and $a > 0$ such that $\lim a_n = a$. Show that

$$|a_n^{1/N} - a^{1/N}| \leq \frac{|a_n - a|}{a^{\frac{N-1}{N}}}$$

and

$$\lim_n a_n^{1/N} = a^{1/N} \quad (*)$$

(iii) Is $(*)$ true if $a = 0$ rather than $a > 0$? Given your reasoning.

Solution. Let $x, y > 0$. Note $x^N - y^N = (x - y)(x^{N-1} + x^{N-2}y + \cdots + y^{N-1})$. Therefore

$$|x^N - y^N| = |x - y|\,|x^{N-1} + x^{N-2}y + \cdots + y^{N-1}| \geq |x - y|\,y^{N-1}$$

[1] please kindly send an email to `nclliu@math.cuhk.edu.hk` if you have spotted any typo/error/mistake.

Hence (i) is proved. The inequality in (ii) is proved by putting $y = a^{1/N}$ and $x = a_n^{1/N}$. By assumption $\lim_n a_n = a$, so $\lim_n |a_n - a| = 0$. Hence $\lim_n a_n^{1/N} - a^{1/N} = 0$.

For (iii), $(*)$ is true if $a = 0$. We prove $\lim_n a_n^{1/N} = 0$. Let $\epsilon > 0$. Then $\epsilon^N > 0$. By definition of $\lim_n a_n = a$, there is M such that $|a_n - 0| < \epsilon^N$ for all $n > M$, i.e. $|a_n^{1/N}| < \epsilon$ for all $n > M$.

Q3. Check by definitions (in terms of ϵ-N that if $x = \lim_n x_n$ and $y = \lim_n y_n > 0$ exists in $\mathbb{R}$, then $\lim_n \frac{x_n}{y_n} = \frac{x}{y}$

Solution. Please refer to **Theorem 3.2.3 (b)** of the textbook (Introduction to Real Analysis 4th edition, Bartle)

Q4. Check by definitions that if $\lim_n z_n = 6$ then

$$\lim_n \frac{z_n^3 + 4}{z_n - 5} = 220$$

Solution. Let $\epsilon > 0$.

Consider $\frac{1}{2} > 0$. By definition there is N_1 such that $|z_n - 6| < \frac{1}{2}$ for all $n > N_1$, so $\frac{1}{2} < z_n - 5$ for all $n > N_1$.

Note

$$\Big|\frac{z_n^3 + 4}{z_n - 5} - 220\Big| = \Big|\frac{(z_n - 6)(z_n^2 + 6z_n - 184)}{z_n - 5}\Big| \leq \frac{|z_n - 6|\,(|z_n|^2 + 6|z_n| + 184)}{|z_n - 5|}$$

Since $(z_n)_{n=1}^{\infty}$ is convergent, it is bounded by some $M > 0$, i.e. $|z_n| \leq M$ for all n. Therefore

$$\Big|\frac{z_n^3 + 4}{z_n - 5} - 220\Big| \leq \frac{|z_n - 6|(M^2 + 6M + 184)}{|z_n - 5|}$$

Consider $\epsilon/2(M^2 + 6M + 184) > 0$. By definition of $\lim_n z_n = 6$ there is N_2 such that $|z_n - 6| \leq \epsilon/2(M^2 + 6M + 184)$ for all $n > N_2$. Now for $n > \max(N_1, N_2) := N$, we have

$$\Big|\frac{z_n^3 + 4}{z_n - 5} - 220\Big| \leq \frac{|z_n - 6|(M^2 + 6M + 184)}{|z_n - 5|} < \epsilon$$

Hence $\lim_n \frac{z_n^3+4}{z_n-5} = 220$.