1.* Let $\sum_{n=1}^{\infty} a_n$ be a positive series, and $\sum_{n=1}^{\infty} 2^n a_{2^n}$ be its "condensed" one and let

$$s_n := \sum_{i=1}^{n} a_i , \qquad \forall n = 1, 2, \ldots$$

$$t_n := \sum_{i=1}^{n} 2^{i} a_{2^i} \qquad \forall n = 1, 2, \ldots$$

Suppose that $(a_n) \downarrow 0$: $\lim_n a_n = 0$ & $a_n \geq a_{n+1}$ $\forall n$.
Show that $\sum_{n=1}^{\infty} a_n < +\infty$ iff $\sum_{n=1}^{\infty} 2^n a_{2^n} < +\infty$
(known as Cauchy Condensation test) along the following steps : show that, $\forall n \in \mathbb{N}$, $n \geq 2$

(i) $s_{2^n - 1} \leq a_1 + t_{n-1}$,

(ii) $s_{2^n} \geq a_1 + \dfrac{t_n}{2}$.

2*. We have learnt that, for any bounded sequence (x_n) :

$$\limsup_n x_n := \lim_n s_n = \max L = \inf E ,$$

where

$$s_n := \sup\{x_m : n \leq m\},$$

$$L := \{l \in \mathbb{R} : \exists \text{ some subseq of } (x_n) \text{ convergent to } l\}$$

$$E := \{u \in \mathbb{R} : \exists \text{ some } N \in \mathbb{N} \text{ s.t. } x_n \leq u \ \forall n \geq N\}$$

Is it true that $\limsup x_n \in E$?
i.e. $\limsup x_n = \min E$?

Substantiate your answer (i.e. prove your assertion or provide a counter-example).

3. Recall that, for $x_0 \in \mathbb{R}$ & $\delta > 0$,

$$V_\delta(x_0) := \{x \in \mathbb{R} : |x - x_0| < \delta\}$$

(so-called the δ-neighborhood of x_0). Check all equalities below (with a non-empty set A of real numbers)

$$\begin{aligned} A^c &:= \{c \in \mathbb{R} : V_\delta(x_0) \text{ intersects } A \setminus \{c\} \ \forall \delta > 0\} \\ &= \{c \in \mathbb{R} : \forall \delta > 0 \ \exists a \in A \text{ s.t. } 0 < |a - c| < \delta\} \\ &= \{c \in \mathbb{R} : \forall n \in \mathbb{N} \ \exists a_n \in A \setminus \{c\} \text{ s.t. } |a_n - c| < \tfrac{1}{n}\} \\ &= \{c \in \mathbb{R} : \exists \text{ a seq } (a_n) \text{ in } A \setminus \{c\} \text{ s.t. } \lim_n a_n = c\} \\ &= \{c \in \mathbb{R} : \operatorname{dist}(c, A \setminus \{c\}) = 0\}, \end{aligned}$$

where $\operatorname{dist}(x, B) := \inf\{|x - b| : b \in B\}$, $\forall$ non-empty set B of real numbers.

4*. Let

$$A := (1, \sqrt{2}) \cap \mathbb{Q}$$

Identify A^c with each of the following methods:

(a) Check via definition given in Q3

(b) Let $f_c(x) := \operatorname{dist}(x, A \setminus \{c\})$ $(\forall x \in \mathbb{R})$.

Determine f_c and hence identify A^c.

5*. Let $x_0 \in A^c$, $f : A \to \mathbb{R}$ and $l_1, l_2 \in \mathbb{R}$. Suppose $f(x) \to l_i$ $(i = 1, 2)$ as $x \to x_0$ $(x \in A)$. Show that $l_1 = l_2$. (Hint: show that $|l_1 - l_2| < \varepsilon$ $\forall \varepsilon > 0$).