Limit of Functions

Definition (c.f. Definition 4.1.4)**.** Let c be a cluster point of $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function. A real number L is said to be a *limit* of f at c, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x \in A$ and $0 < |x - c| < \delta$,

$$|f(x) - L| < \varepsilon.$$

In this case, f is said to *converge* to L at c and we denote

$$L = \lim_{x \to c} f(x).$$

Remark. We can only discuss the limit of a function at cluster points of its domain. For example, if f is a function defined on $A = \{1/n : n \in \mathbb{N}\}$, then we can only talk about the limit of f at 0. Also, if a function converges at a point, then the limit is unique.

Example 1 (c.f. Section 4.1, Ex.10(b))**.** Use the definition of limit to show that

$$\lim_{x \to -1} \frac{x+5}{2x+3} = 4.$$

Solution. Note that

$$\left| \frac{x+5}{2x+3} - 4 \right| = \left| \frac{(x+5) - 4(2x+3)}{2x+3} \right| = \frac{7}{|2x+3|}|x+1|, \quad \forall x \in \mathbb{R} \setminus \{-1.5\}.$$

Also if $|x+1| < 0.25$, then $-1.25 < x < -0.75$. Hence $0.5 < 2x + 3 < 1.5$. In this case,

$$\left| \frac{x+5}{2x+3} - 4 \right| = \frac{7}{|2x+3|}|x+1| < \frac{7}{0.5}|x+1| = 14|x+1|.$$

Let $\varepsilon > 0$ and take $\delta = \min\{0.25, \varepsilon/14\}$. Then whenever $0 < |x+1| < \delta$,

$$\left| \frac{x+5}{2x+3} - 4 \right| < 14|x+1| < 14\delta \leq \varepsilon.$$

The result follows by definition.

Exercise. Find $\lim_{x \to 2} \frac{x^3 - 4}{x^2 + 1}$ and prove your assertion.

We can find a sequence in $A \subseteq \mathbb{R}$ to approximate its cluster point c. From this, we deduce the **Sequential Criteria** of limits.

Theorem (c.f. Theorem 4.1.2)**.** *Let $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$. c is a cluster point of A if and only if there exists a sequence (a_n) in A such that $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.*

Theorem (c.f. Theorem 4.1.8)**.** *Let c be a cluster point of $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function. Let $L \in \mathbb{R}$. The following are equivalent:*

(i) $\lim_{x\to c} f(x) = L$.

(ii) For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.

Divergence Criteria (c.f. 4.1.9)**.** *Let c be a cluster point of $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function.*

(a) If $L \in \mathbb{R}$, then f does not have a limit L at c if and only of there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does not converge to L.

(b) The function f does not have a limit at c if and only of there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does not converge in $\mathbb{R}$.

Example 2 (c.f. Section 4.1, Ex.15)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) = x$ if x is rational, and $f(x) = 0$ if x is irrational.

(a) Show that f has a limit at $x = 0$.

(b) Use a sequential argument to show that if $c \neq 0$, then f does not have a limit at c.

Solution. For (a), notice that we always have $|f(x)| \leq |x|$ because $f(x)$ equals either x or 0. Let $\varepsilon > 0$ and take $\delta = \varepsilon$. Then whenever $0 < |x| < \delta$, we have

$$|f(x) - 0| \leq |x| < \delta = \varepsilon.$$

The result follows by definition. For (b), we need to find a sequence (x_n) of real numbers that converges to c with $x_n \neq c$ for all $n \in \mathbb{N}$ and $(f(x_n))$ is divergent. For each $n \in \mathbb{N}$, consider the interval $(c, c + 1/n)$. By the density of rational and irrational numbers, we can find some rational number y_n and irrational number z_n in $(c, c + 1/n)$. Define

$$x_n = \begin{cases} y_n, & \text{if } n \text{ is odd,} \\ z_n, & \text{if } n \text{ is even.} \end{cases}$$

Then we have

$$f(x_n) = \begin{cases} x_n, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

By **Squeeze Theorem**, we have $\lim x_n = c$. Also, by considering the odd and even subsequence of $(f(x_n))$, we see that $f(x_n)$ is divergent. It follows by the **Divergence Criteria** that f does not have a limit at c.

Exercise. Prove that $(f(x_n))$ is divergent.

One-sided Limits and Limits Involving Infinity

Definition (c.f. Definition 4.3.1)**.** Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. Suppose $c \in \mathbb{R}$ is a cluster point of $A \cap (-\infty, c)$. Then $L \in \mathbb{R}$ is said to be a *left-hand limit* of f at c if for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x \in A$ and $0 < c - x < \delta$,

$$|f(x) - L| < \varepsilon.$$

In this case, we write $\lim_{x \to c-} f(x) = L$.

Exercise. Formulate the definition for the *right-hand limit* $\lim_{x \to c+} f(x) = L$.

The following theorem is why we like to consider one-sided limits instead of usual limits. Especially when we deal with piecewise defined functions.

Theorem (c.f. Theorem 4.3.3)**.** *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. Suppose $c \in \mathbb{R}$ is a cluster point of both of the sets $A \cap (c, \infty)$ and $A \cap (-\infty, c)$. Then $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c-} f(x) = L = \lim_{x \to c+} f(x)$.*

Definition (c.f. Definition 4.3.5)**.** Let c be a cluster point of $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. f is said to *tend to ∞ as $x \to c$* if for any $\alpha \in \mathbb{R}$, there exists $\delta > 0$ such that whenever $x \in A$ and $0 < |x - c| < \delta$,

$$f(x) > \alpha.$$

In this case, we write $\lim_{x \to c} f(x) = \infty$.

Definition (c.f. Definition 4.3.10)**.** Let $A \subseteq \mathbb{R}$ with $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function. $L \in \mathbb{R}$ is said to be a *limit of f as $x \to \infty$* if for any $\varepsilon > 0$, there exists $K > a$ such that

$$|f(x) - L| < \varepsilon, \quad \forall x > K.$$

In this case, we write $\lim_{x \to \infty} f(x) = L$.

Remark. The assumption $(a, \infty) \subseteq A$ can be relaxed to A being not bounded above.

Exercise. Formulate the definitions for the following:

(a) $\lim_{x \to c} f(x) = -\infty$

(b) $\lim_{x \to c+} f(x) = \infty$

(c) $\lim_{x \to c+} f(x) = -\infty$

(d) $\lim_{x \to c-} f(x) = \infty$

(e) $\lim_{x \to c-} f(x) = -\infty$

(f) $\lim_{x \to -\infty} f(x) = L$

(g) $\lim_{x \to \infty} f(x) = \infty$

(h) $\lim_{x \to -\infty} f(x) = \infty$

(i) $\lim_{x \to \infty} f(x) = -\infty$

(j) $\lim_{x \to -\infty} f(x) = -\infty$

Example 3. Find $\lim_{x\to\infty} \dfrac{2x^2+x+1}{x^2+3}$ and prove your assertion.

Solution. By high school limit calculation, we can see that the limit is 2. The prove is very similar to the limit of sequence. Note that

$$\left|\frac{2x^2+x+1}{x^2+3}-2\right| = \frac{|x-5|}{x^2+3} = \frac{|x-5|}{(x-5)^2+10x-22}, \quad \forall x \in \mathbb{R}.$$

Also if $x > 5$, we have $10x - 22 > 28 > 0$. In this case,

$$\left|\frac{2x^2+x+1}{x^2+3}-2\right| = \frac{|x-5|}{(x-5)^2+10x-22} < \frac{|x-5|}{(x-5)^2+0} = \frac{1}{x-5}.$$

Let $\varepsilon > 0$. Take $K > \max\{5, 1/\varepsilon + 5\}$. Then whenever $x > K$,

$$\left|\frac{2x^2+x+1}{x^2+3}-2\right| < \frac{1}{x-5} < \frac{1}{K-5} < \varepsilon.$$

It follows by definition that $\lim_{x\to\infty} \dfrac{2x^2+x+1}{x^2+3} = 2$.

Exercise. Find $\lim_{x\to-\infty} \dfrac{2x^2-1}{x^2+x+3}$ and prove your assertion.

Example 4 (c.f. Section 4.3, Ex.8)**.** Let $f : (0,\infty) \to \mathbb{R}$ be a function. Prove that $\lim_{x\to\infty} f(x) = L$ if and only if $\lim_{x\to 0+} f(1/x) = L$.

Solution. ($\Rightarrow$) Suppose $\lim_{x\to\infty} f(x) = L$. Let $\varepsilon > 0$. Then there exists $K > 0$ such that

$$|f(x) - L| < \varepsilon, \quad \text{whenever } x > K. \tag{1}$$

Take $\delta = 1/K > 0$. Then whenever $0 < x < \delta$, we have $1/x > K$. Hence by (1),

$$|f(1/x) - L| < \varepsilon.$$

($\Leftarrow$) Suppose $\lim_{x\to 0+} f(1/x) = L$. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$|f(1/x) - L| < \varepsilon, \quad \text{whenever } 0 < x < \delta. \tag{2}$$

Take $K = 1/\delta > 0$. Then whenever $x > K$, we have $0 < 1/x < \delta$. Hence by (2),

$$|f(1/(1/x)) - L| = |f(x) - L| < \varepsilon.$$