General Information

- Textbook: *Introduction to Real Analysis* by Robert G. Bartle, Donald R. Sherbert. (Try to google the title of the textbook for MORE information!) This textbook will also be used in the next course MATH2060 Mathematical Analysis II.
- Please visit the course web-page at `https://www.math.cuhk.edu.hk/course/2021/math2050a` **frequently** to get the most updated information. It shall contain the information for homework and tests, as well as lecture notes and tutorial notes.
- I am the tutor of this section. You may call me **Ernest**. You are welcomed to send me an email if you need help. My email address is **ylfan@math.cuhk.edu.hk**.
- The grader of this section is **Lam Ka Lok**. He is responsible for grading the assignments. His email address is **kllam@math.cuhk.edu.hk**
- Please submit your homework in **one PDF file** to **Blackboard**.

Supremum and Infimum

This course focuses on the number line $\mathbb{R}$. The following definitions are fundamental and important for analysis.

Definition (c.f. Definition 2.3.1)**.** Let X be a **non-empty** subset of $\mathbb{R}$.

- A number $u \in \mathbb{R}$ is said to be an *upper bound* of X if $x \leq u$ for all $x \in X$.
- A number $\ell \in \mathbb{R}$ is said to be a *lower bound* of X if $x \geq \ell$ for all $x \in X$.
- X is said to be *bounded above* if it has an upper bound.
- X is said to be *bounded below* if it has a lower bound.
- X is said to be *bounded* if it is bounded above and bounded below.
- X is said to be *unbounded* if it is not bounded.

Definition (c.f. Definition 2.3.2)**.** Let X be a **non-empty** subset of $\mathbb{R}$.

- The *supremum* of X, denoted by $\sup X$, is defined as the **least upper bound** of X. i.e. $\sup X \geq x$ for all $x \in X$ and $\sup X \leq u$ whenever u is an upper bound of X.
- The *infimum* of X, denoted by $\inf X$, is defined as the **greatest lower bound** of X. i.e. $\inf X \leq x$ for all $x \in X$ and $\inf X \geq \ell$ whenever ℓ is a lower bound of X.

Remark. Try to understand the definitions by visualising different non-empty subsets of $\mathbb{R}$.

There are **two steps** to show that a number α is the supremum (resp. infimum) of a non-empty subset $X \subseteq \mathbb{R}$. We first need to show that α is an upper (resp. lower) bound of X. Then, we need to show that α is the least (resp. greatest) among all possible upper (resp. lower) bounds of X. Observe the following example.

Example 1. Let $X = [0, 1]$. Show that $\inf X = 0$.

Solution. We need to check:

(i) 0 is a lower bound of X. i.e., $x \geq 0$ for all $x \in X$.

(ii) 0 is the greatest lower bound of X. i.e., $0 \geq \ell$ whenever ℓ is a lower bound of X.

Statement (i) is trivial. It remains to check statement (ii). Let ℓ be any lower bound of X. By the definition of lower bound, $x \geq \ell$ for all $x \in X$. If we take $x = 0$, then $0 \geq \ell$.

Exercise. Show that $\sup X = 1$.

You may find it difficult to do the same thing to the set $Y = (0, 1)$ as in the previous example. The following lemma gives an alternative method to determine whether a given upper bound is the least among all possible upper bounds.

Lemma (c.f. Lemma 2.3.3 and Lemma 2.3.4)**.** *Let $u \in \mathbb{R}$ be an upper bound of a non-empty subset X of $\mathbb{R}$. The following statements are equivalent:*

(i) u is the supremum of X, i.e. $u = \sup X$.

(ii) If $v < u$, then there exists $x \in X$ such that $v < x$.

(iii) For every $\varepsilon > 0$, there exists $x \in X$ such that $u - \varepsilon < x$.

Remark. Try to formulate and prove the infimum version of this lemma.

Example 2. Let $Y = (0, 1)$. Show that $\sup Y = 1$.

Solution. It is trivial that 1 is an upper bound of Y. Here we provide two proofs by checking statements (ii) and (iii) above.

(ii) Let $v < 1$. Then we can find a number $x \in \mathbb{R}$ such that $\max\{0, v\} < x < 1$. It follows that $x \in Y$ and $v < x$.

(iii) Let $\varepsilon > 0$. Then we can find a number $x \in \mathbb{R}$ such that $\max\{0, 1 - \varepsilon\} < x < 1$. It follows that $x \in Y$ and $1 - \varepsilon < x$.

Exercise. Show that $\inf Y = 0$.

The Completeness Property of $\mathbb{R}$

The most important property of the real number system $\mathbb{R}$ is stated below:

The Completeness Property of $\mathbb{R}$ (c.f. 2.3.6)**.** *Every bounded above non-empty subset of $\mathbb{R}$ has a supremum in $\mathbb{R}$.*

Archimedean Property (c.f. 2.4.3)**.** *If $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x \leq n$.*

Corollary (c.f. Corollary 2.4.5)**.** *If $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $0 < 1/n < \varepsilon$.*

Let's do some examples to see how **Archimedean Property** is applied.

Example 3 (c.f. Section 2.4, Ex.1)**.** Show that $\sup\{1 - 1/n : n \in \mathbb{N}\} = 1$.

Solution. We need to check that:

- 1 is an upper bound of the set.

  Proof. Let $n \in \mathbb{N}$. Since $1/n \geq 0$, we have $1 - 1/n \leq 1 - 0 = 1$. □

- $1 \leq u$ whenever u is an upper bound of the set.

  Proof. Suppose on a contrary that there is an upper bound $u \in \mathbb{R}$ of the set with $u < 1$. Apply **Archimedean Property** on the number $\varepsilon = 1 - u > 0$. There exists $n \in \mathbb{N}$ such that $0 < 1/n < 1 - u$. It follows that

  $$u < 1 - \frac{1}{n},$$

  which contradicts to the fact that u is an upper bound of the set. □

The following famous theorem is also an application of the **Archimedean Property**. It describes how rational numbers and irrational numbers are distributed in $\mathbb{R}$.

The Density Theorem (c.f. 2.4.8)**.** *If x and y are any real numbers with $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.*

Now let's do more examples to get yourselves familiar with the materials.

Example 4. Determine the supremum and infimum of the following sets (if such exist):

(a) $X_1 = (0, 1]$

(b) $X_2 = \mathbb{N}$

(c) $X_3 = [0, 1) \cup \{2\}$

(d) $X_4 = (0, 1) \cap \mathbb{Q}$

(e) $X_5 = \mathbb{R} \setminus [-1, 1]$

(f) $X_6 = \{n + 1/n : n \in \mathbb{N}\}$

Solution. Visualising the given sets helps to determine the answers.

(a) $\inf X_1 = 0$; $\sup X_1 = 1$.

(b) $\inf X_2 = 1$; $\sup X_2$ does not exist.

(c) $\inf X_3 = 0$; $\sup X_3 = 2$.

(d) $\inf X_4 = 0$; $\sup X_4 = 1$.

(e) Both of $\inf X_5$ and $\sup X_5$ does not exist.

(f) $\inf X_6 = 2$; $\sup X_6$ does not exist.

Exercise. Prove the above results.

Example 5 (c.f. Section 2.3, Ex.11)**.** Let A, B be bounded non-empty subsets of $\mathbb{R}$ with $A \subseteq B$. Show that $\inf B \leq \inf A \leq \sup A \leq \sup B$.

Solution. Here we need to check three inequalities. I shall check the first two and leave the remaining one as an **Exercise**.

- $\inf A \leq \sup A$: Since A is non-empty, pick any $a \in A$. Since the infimum and supremum of A is in fact a lower bound and an upper bound of A respectively, we have

$$\inf A \leq a \leq \sup A.$$

- $\inf B \leq \inf A$: Pick any $a \in A$. Since $A \subseteq B$, $a \in B$. Hence we have $\inf B \leq a$. Notice that the choice of $a \in A$ is arbitrary, this implies that $\inf B$ is a lower bound of A. As the infimum is the greatest lower bound, we have $\inf B \leq \inf A$.

Example 6. Let S be a bounded non-empty subset of $\mathbb{R}$. Show that

$$\inf S = -\sup(-S).$$

Solution. We often prove that $x = y$ by checking $x \leq y$ and $x \geq y$ in analysis problems.

- $\inf S \leq -\sup(-S)$.

  Proof. Let $s \in S$. Notice that $\inf S \leq s$. This implies that

$$-\inf S \geq -s, \quad \forall s \in S.$$

  i.e., $-\inf S$ is an upper bound of the set $-S$. Hence $-\inf S \geq \sup(-S)$. It follows that $\inf S \leq -\sup(-S)$. □

- $-\sup(-S) \leq \inf S$.

  Proof. Let $s \in S$. Since $-s \in -S$, we have $-s \leq \sup(-S)$. This implies that

$$s \geq -\sup(-S), \quad \forall s \in S.$$

  i.e., $-\sup(-S)$ is a lower bound of S. It follows that $-\sup(-S) \leq \inf S$. □