MATH 2050A - HW 9 - Solutions

Commonly missed steps in Purple

Solutions

1 (P.148 Q8)**.** Let f, g be real-valued, uniformly continuous functions on $\mathbb{R}$. Show that the composition $f \circ g$ is uniformly continuous on $\mathbb{R}$

Solution. Let $\epsilon > 0$. Then by uniform continuity of f, there exists η such that $|f(x) - g(y)| < \epsilon$ for all $x, y \in \mathbb{R}$ with $|x - y| < \eta$. Then by uniform continuity of g, there exists $\delta > 0$ such that $|g(x) - g(y)| < \eta$ for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$.
Therefore, when $x, y \in \mathbb{R}$ with $|x - y| < \delta$, we have $|g(x) - g(y)| < \eta$ where $g(x), g(y) \in \mathbb{R}$ and so $|f(g(x)) - f(g(y))| < \epsilon$, that is $|f \circ g(x) - f \circ g(x)| < \epsilon$. The result follows by the definition of uniform continuity.

2 (P.148 Q10)**.** Let $A \subset \mathbb{R}$ be a bounded subset. Suppose f is a real-valued function uniformly continuous on A. Show that f is bounded on A.

Solution.
Method 1: Proof by Contradiction
Suppose f were not bounded on A. Then there exists a sequence (x_n) in A such that $|f(x_n)| \geq n$ (why?). Since A is bounded, then its closure $\overline{A}$ is bounded[1]. Therefore the closed and bounded set $\overline{A}$ is compact. As (x_n) is a sequence in $A \subset \overline{A}$, it follows from the (sequential) definition of a compact set that there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x$ for some $x \in \overline{A}$[2]. In particular, since (x_{n_k}) converges, it is a Cauchy sequence in A. By uniform continuity of f, $(f(x_{n_k}))$ is a Cauchy sequence and therefore is bounded. Nonetheless, by the assumption $|f(x_{n_k})| \geq n_k$ for all $k \in \mathbb{N}$ and so the sequence is unbounded. Therefore contradiction arises. It must be the case that A is bounded.
Method 2: Direct Proof
Since f is uniformly continuous on A, there exists $\delta > 0$ such that $|f(x) - f(y)| < 1$ whenever $|x - y < \delta|$ and $x, y \in A$.
Next we show that A can be covered by a finite union of open intervals with radius $\delta/2$[3]: note that as in Method 1, $\overline{A}$ is compact. By the Heine-Borel Property, the open cover $\overline{A} \subset \bigcup_{a \in A} B(a, \delta/2)$ (why is this, which runs over $a \in A$ instead of $\overline{A}$, an open cover?) admits a finite subcover. Hence, there exists $a_1, \ldots, a_N \in A$ where $N \in \mathbb{N}$ such that $A \subset \overline{A} \subset \bigcup_{i=1}^{N} B(a_i, \delta/2)$.
Now take $M := \max\{|f(x_i)|\}_{i=1}^{N}$. Finally, let $a \in A$, there exists $1 \leq i \leq N$ such that $|a - a_i| < \delta/2$. By the definition of δ, we have by the triangle inequality that

$$|f(a)| \leq |f(a) - f(a_i)| + |f(a_i)| \leq 1 + M$$

It follows that f is bounded by $1 + M$ where M is independent of the choice of $a \in A$.

[1] In fact if $M > 0$ is a bound for A, it is also a bound of $\overline{A}$. The proof is as follows: Fix $x \in \overline{A}$. Let $\epsilon > 0$. Then there exists $a \in A$ such that $|x - a| < \epsilon$. By triangle inequality, $|x| < \epsilon + |a| \leq \epsilon + M$. It follows that $|x| \leq M$ as $\epsilon \to 0$. This shows that M is a bound for $\overline{A}$. In fact we can show further that $\sup |A| = \sup |\overline{A}|$ similarly. You may use this fact without proof.

[2] Alternatively, you can apply the Bolzano-Weierstrauss Theorem on the bounded sequence (x_n) directly without considering compact sets. However, having a basic understanding of compact sets, you should be seeing that these "two" proofs are really two sides of the same coin.

[3] This property is the so-called *totally boundedness*. A subset $A \subset \mathbb{R}$ is totally bounded if for all $\epsilon > 0$, A can be covered by a finite number of $\epsilon-$ balls. The Heine-Borel Property tells us that a subset of $\mathbb{R}$ is bounded if and only if totally bounded. We leave the proof of the above fact as exercise.