MATH 2050A - HW 3 - Comments and Common Mistakes:

Common Misktakes:

Question 1:

Let $y_n := \sqrt{n+1} - \sqrt{n}$ for all $n \in \mathbb{N}$. Show that $(\sqrt{n}y_n)$ converges and find the limit.

1. There are mainly two approaches to this question: using algebraic property of limit and checking against definition directly. Using the algebraic approach, MANY of you have written something like

   Hence, $\lim(\sqrt{n}y_n)$
   $$= \lim\left(\frac{1}{\sqrt{1+\frac{1}{n}}+1}\right) = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$
   Therefore, $(\sqrt{n}y_n)$ converges.

   The reasoning here is reversed: when you distribute the limit into individual terms, you a priori (預先) assumed the the limit $(\lim \sqrt{n}y_n)$ had existed. The conclusion here doesn't make any sense. Indeed what the algebraic properties says is that if the limit of every individual sequence exists, then there are two consequences: (a) the limit of the sum/product/etc exists; (b) you can compute it by distributing the limit operation into individual terms. These two consequences come together from the fact that individual limits exist but there is no implications between them.

   In fact, whenever you compute a limit, or use the notation $\lim x_n$ for a sequence x_n, you a priori assumed limit of individual terms exists and hence the existence of the limit by the Algebraic properties of limit. It is nonsense to have the computational formula if the limit does not exist, or to say the limit exists because you can compute it. Delete the last line to correct it.

2. For students who prove this by definition, they often write like the following. Please think about what the problem is.

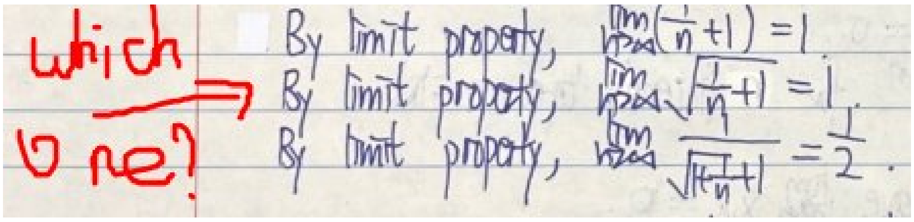
   For the sequence x_n.
   $x_n = \frac{1}{2}\cdot\sqrt{\frac{n}{n+1}} = \frac{1}{2}\sqrt{1-\frac{1}{n+1}}$
   Let $\varepsilon > 0$, $L = \frac{1}{2}$
   $|x_n - L| = \frac{1}{2}\left(1-\sqrt{1-\frac{1}{n+1}}\right) < \varepsilon$
   Then $1-\frac{1}{n+1} > 1-\frac{1}{n} > (1-2\varepsilon)^2$
   $\frac{1}{n} < 1-(1-2\varepsilon)^2 = 4\varepsilon(1-\varepsilon)$
   Hence we get $n > \frac{1}{4\varepsilon(1-\varepsilon)}$
   Then we get that for any $\varepsilon > 0$, there exists $K > \frac{1}{4\varepsilon(1-\varepsilon)}$ $\in \mathbb{N}$
   $|x_n - \frac{1}{2}| < \varepsilon$ for any $n \geq K$
   We could get $\lim x_n = \frac{1}{2}$

   To use the $\epsilon - N$ definition, we needs to find $N(\epsilon)$ for every $\epsilon > 0$ such that for sufficient large

n $(n \geq N)$, we have $|x_n - L| < \epsilon$. The main point of proving via the definition is to find such $N(\epsilon)$ for every ϵ, or is to find *when* the sequence will come close to the limit. To do this, you often first consider what $N(\epsilon)$ would be *sufficient*. Therefore, we consider $|x_n - L| < \epsilon$ as if such $N(\epsilon)$ exists to find out how large we need.

Nonetheless, finding out what $N(\epsilon)$ is enough does nothing to prove the convergence of the statement; what we need is to find $N(\epsilon)$, *not* what is enough for $N(\epsilon)$. (i.e. You have "If A then B", but it doesn't mean A is true). To ensure such $N(\epsilon)$ *really exists*, you have to use the Archimedean Property.

3. If you are using the Algebraic Approach, please show why the limit operation can commute with square root, that is, why $x_n \to x$ implies $\sqrt{x_n} \to \sqrt{x}$. We have not included this in the Lecture Note and the Tutorials.



Once again, you may have learn many facts about Limits in High School. Please forget ALL of them here in this course. Keep in mind that we are *rebuilding the entire theory from axioms*! Some facts you have learnt are trivial and some are not. There is only 1 rule here: if you want to use something we have not taught (in Lecture Notes or in Tutorials): justify it.

4. Some students used the Bounded Convergence Theorem to show the limit of $\sqrt{n}y_n$ exists and after that, used the $\epsilon - N$ definition to prove the limit. The use of the Bounded Convergence Theorem is redundant; you prove the existence of a limit and find its value if at the same time if you use the $\epsilon - N$ definition.

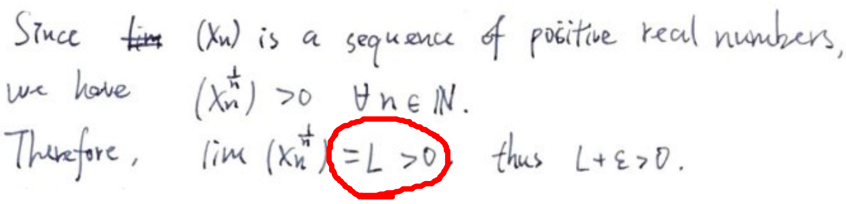
Question 2:

Let (x_n) be a sequence of positive real numbers such that $L := \lim(x_n^{1/n}) < 1$.

i). Show that there exists a real number $r \in (0, 1)$ such that $x_n \in (0, r^n)$ for all sufficiently large $n \in \mathbb{N}$.

ii). Hence, show that $\lim x_n = 0$.

1. This question basically tests you the order properties of limit and the squeeze theorem. Below are some standard common mistakes.



It is important: LIMIT DOES NOT PRESERVE STRICT INEQUALITY. An easy example would be to consider the sequence (n^{-1}) in which all terms are > 0 but the limit is 0. Limit only preserve partial inequalities $\leq$ and $\geq$.

Nonetheless, there is something you can talk about with limit and strict inequality: if $\lim_n x_n < K$ for some $K \in \mathbb{R}$, then $x_n < K$ for large n. Please see Lemma 0.1 in the solution HW3 for details.

2. This is a standard mistake concerning Squeeze Theorem

(ii) Note that $0 \le X_n \le r^n$ from (i),
Since $0<r<1$, we have $\lim(r^n) = 0$, $\lim(0) \le \lim(X_n) \le 0$
By Squeeze Theorem, $\lim(X_n) = 0$

Keep in mind that whenever you write $\lim_n x_n$, you a priori (預先) assumed the existence of the limit. What Squeeze Theorem tells is that if you have three sequences satisfying $a_n \le x_n \le b_n$ with a_n, b_n convergent and converging to the same limit, say L, the there are *two* consequences: (a) (x_n) converges, i.e., $\lim_n x_n$ exists. (b) you can compute it by $\lim_n x_n = \lim_n a_n = \lim_n b_n = L$. You never use the notation $\lim_n x_n$ before applying Squeeze Theorem, or before showing $\lim_n x_n$ exists. Use the Squeeze Theorem to claim the existence of limit instead.

3. To prove $\lim_n r_n = 0$ where $0 < r < 1$, many students used the logarithmic function $x \mapsto \ln(x)$. Some students also used the floor function $x \mapsto \lfloor x \rfloor$.

what is ln?
what is [ ]?

Since $x_n > 0$, $\forall n > N$. the $\lim_{n\to v} x_n \geq 0$.
there exists $k = [\frac{\ln\varepsilon}{\ln r}] + 1$, then $k = [\frac{\ln\varepsilon}{\ln r}] + 1 > \frac{\ln\varepsilon}{\ln r}$
Note that $r^{n+1} - r^n = r^n(r-1) < 0$, hence (r^n) is strictly decreasing.
therefore, $|x_n - 0| < |r^n - 0| = r^n < r^k < r^{\frac{\ln\varepsilon}{\ln r}} = r^{\log_r \varepsilon} = \varepsilon$
thus $\forall \varepsilon > 0$. $\exists\, k = [\frac{\ln\varepsilon}{\ln r}] + 1$ s.t. $|x_n - 0| < \varepsilon$. $n > k$
Thus $\lim_{n\to\infty} x_n = 0$

Once again, we are *rebuilding the entire theory from axioms.* Please use only facts/definitions we have taught. If you want to use something we have not taught (in Lecture Notes or in Tutorials): justify it. I did not deduct marks from you this time.

As a matter of fact, using the Binomial Theorem and the Archimedean Principle can replace the use of the logarithmic function and the floor function. Please see Lemma 0.2 in Solution HW3 for details.

Question 3:

Let (x_n) be a covergent sequence of real numbers and (y_n) be such that for all $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that $|x_n - y_n| < \epsilon$ for all $n \geq M$. Does it follow that (y_n) is convergent?

1. Please compare the two solutions below and think about which argument is nonsense.

22) Since (x_n) is convergent, let $\lim(x_n) = L$, then we have $\forall \varepsilon > 0, \exists n \in \mathbb{N}$ s.t. $|x_n - L| < \varepsilon \ \forall n \geq N$.
Also we have $\forall \varepsilon > 0, \exists M \in \mathbb{N}$ s.t. $|x_n - y_n| < \varepsilon \ \forall n \geq M$.
Then $\forall \varepsilon > 0$, take $K = \max\{M, N\}$, we have
$$|y_n - L| = |y_n - x_n + x_n - L| \leq |y_n - x_n| + |x_n - L| < \varepsilon + \varepsilon = 2\varepsilon \quad \forall n \geq K.$$
Hence (y_n) is convergent. #

Arguement from Student A

3. Yes. (y_n) is also convergent.

proof: Let $\ell = \lim_{n\to\infty} x_n$. Let $\varepsilon > 0$ be given.

Since $\frac{1}{2}\varepsilon > 0$, $\exists M \in \mathbb{N}$ such that $|x_n - y_n| < \frac{1}{2}\varepsilon$ for all $n \geq M$.

Since $\ell = \lim_{n\to\infty} x_n$, $\exists m \in \mathbb{N}$ such that $|x_n - \ell| < \frac{1}{2}\varepsilon$ for all $n > m$.

Hence for $n \in \mathbb{N}$, $n > m + M$, by triangle inequality, we have

$$|y_n - \ell| = |y_n - x_n + x_n - \ell| \leq |y_n - x_n| + |x_n - \ell| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Then by definition, (y_n) is also convergent, and $\lim_{n\to\infty} y_n = \ell$. □

Arguement from Student B

The argument from Student A is a typical argument from most of you, which is not ok while the argument from Student B is what you should work towards: the mistake of Student A is that s/he confuses the usage of conditional statements $(\forall x, P(x))$.

Instead of declaring $\epsilon > 0$, Student A starts the proof by repeating the $\epsilon - N$ definition of convergence of sequence and the conditional statement $\forall \epsilon > 0, \exists M \in \mathbb{N}, \ldots$ in the question. S/he then declare the $\epsilon > 0$, and write $K = \max\{M, N\}$ *without* claiming the existence of M, N.

In a conditional statement, ALL variables are *local* to the statement. You cannot use them outside. That says, the first appearance of N and M are local to the conditional statements for all $\epsilon > 0$, there exists .... They are *meaningless* outside them. So, in his argument, $K = \max\{M, N\}$ is undefined because M, N have only appeared before *locally* in the conditional statements.

As a rule of thumb, what you should do is NOT to re-state any conditional statements but to *apply* them to guarantee the existence of some objects.

In the second argument, Student B first declared the variable $\epsilon > 0$, and then he *applies* (without restating) the conditional statements so that the existence of some M is possible. The variables N, M are well-defined *throughout* the entire argument.

To conclude, what Student A needs to do is to define what M, N are before taking $K = \{M, N\}$. This can be accomplished by simply deleting all $\forall \epsilon > 0$ and write *Let* $\epsilon > 0$ in the beginning so that s/he is using the conditional statements instead of restating them. S/he can also insert a declaration of M, N after the last $\forall \epsilon > 0$.

General Comments:

1. Q1 carries 3 marks; Q2 carries 4 marks; Q3 carries 2 marks; 1 mark is for presentation and effort.

2. Most of you have changed the filename. Thank you for your attention.

3. Concerning what theorems you can use to tackle your Homework, you are reminded that in this course we are *rebuilding an entire theory for* $\mathbb{R}$ *from axioms*, which includes the Axiom of Completeness. Therefore, even though you may have learn many facts about limits in High School, please forget ALL of them here in this course. Some facts you have learnt are trivial, but most are not. You are expected to be equipped with the tools (theorems) we give you in Lectures (Notes) and Tutorials. If you want to use something we have not taught (in Lecture Notes or in Tutorials), there is only 1 requirement: justify it.