Week 3

MATH 2040B

September 29, 2020

1 Concepts

☆ V W are both vector space

1. $T : V \to W$ is a linear transformation if

(a) $T(v_1 + v_2) = T(v_1) + T(v_2)$

(b) $T(\alpha v_1) = \alpha T(v_1)$

$T(W) \subset W$

2. Let $T : V \to V$ be linear, then a subspace $W \subset V$ is said to be T-invariant if $T(W) \subset W$. (We will use it later.)

2 Notations

$T: V \to W$

1. $N(T) := \{\vec{x} \in V : T(\vec{x}) = \vec{0}_W\} \subset V$
2. $R(T) := \{T(\vec{x}) : \vec{x} \in V\} \subset W$
3. $\text{Nullity}(T) = \dim N(T), \quad \text{Rank}(T) = \dim R(T)$

3 Formula

1. $\text{Nullity}(T) + \text{Rank}(T) = \dim V$
2. Two facts:

(a) T is injective $\Leftrightarrow N(T) = 0 \Leftrightarrow \text{Nullity}(T) = 0$.
(b) T is surjective $\Leftrightarrow R(T) = W \Leftrightarrow \text{Rank}(T) = \dim W$.

4 Problems

1. Let $T : V \to W$ be a linear transfomation, and assume that $\dim V = \dim W$. Prove that

$$\text{T is injective } \Leftrightarrow T \text{ is surjective}$$

proof: ① $\text{Nullity}(T) + \text{Rank}(T) = \dim V = \dim W$.

T is injective $\Leftrightarrow \text{Nullity}(T) = 0$

$\Leftrightarrow \text{Rank}(T) = \dim V$
$= \dim W$

$\Leftrightarrow T$ is surjective

b) we can determine ~~the~~ $N(T)$.

$N(T) = \{\vec{0}\}$

$T(a,b) = 0$

$\Leftrightarrow (a+b, 0, 2a-b) = 0$

$\Rightarrow \begin{cases} a+b=0 \\ 2a-b=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \end{cases}$

T is injective

we assume T is surjective

c) $T: R^2 \to R^3$

$\Rightarrow \dim R^3 = \operatorname{rank}(T)$

$= \dim R^2 - 0$

$= \dim R^2$

$\dim R^3 \neq \dim R^2$

T is not surjective

2. Let $T:\mathbb{R}^2 \to \mathbb{R}^3$ be given by $T(a,b) = (a+b, 0, 2a-b)$
   a) Show that it is a linear transformation.
   b) Is T injective?
   c) Is T surjective?

a) ① $T(v_1+v_2) = T(v_1) + T(v_2)$

② $T(av_1) = aT(v_1)$.

① we can have ~~two~~ (a_1,b_1), (a_2,b_2)

$T((a_1,b_1)+(a_2,b_2)) = T(a_1+a_2, b_1+b_2)$

$= (a_1+a_2+b_1+b_2, 0, 2a_1+2a_2-b_1-b_2)$

$= (a_1+b_1, 0, 2a_1-b_1) + (a_2+b_2, 0, 2a_2-b_2)$

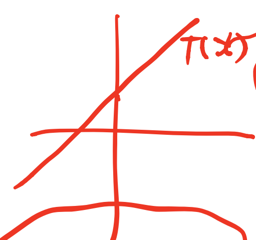
$= T(a_1,b_1) + T(a_2,b_2)$

② $T(\alpha(a_1,b_1)) = T((\alpha a_1, \alpha b_1)) = (\alpha a_1+\alpha b_1, 0, 2\alpha a_1 - \alpha b_1)$

$= \alpha(a_1+b_1, 0, 2a_1-b_1)$

$= \alpha \cdot T(a_1,b_1)$

3. Let $T:\mathbb{R}\to\mathbb{R}$ be given by $T(x) = x+1$, is it linear?



① $T(1+1) = (1+1)+1 = 3$

② $T(1)+T(1) = 1+1+1+1 = 4$

$T(1+1) \neq T(1)+T(1)$

$T(2\cdot 1) = 2\cdot 1+1 = 3$

$\neq$

$2\cdot T(1) = 4$

Mistake

4. Let $T:\mathbb{R}^2\to\mathbb{R}^2$ with $T(1,2) = (3,4)$, $T(3,4) = (1,2)$, what is $T(1,0)$?

$T(1,0)$?

we have to add the property below

proof: $(1,0) = \alpha(1,2) + \beta(3,4)$

This gives a system of linear equations.

$\begin{cases} \alpha + 3\beta = 1 \\ 2\alpha + 4\beta = 0 \end{cases} \Rightarrow \begin{cases} \alpha = -2 \\ \beta = 1 \end{cases}$

$(1,0) = -2(1,2) + 1\cdot(3,4)$

~~we~~ an assumption that T is linear transformation?

$T(1,0) = -2\cdot T(1,2) + 1\cdot T(3,4)$

$= -2(3,4) + 1\cdot(1,2)$

$\boxed{T(v_1+v_2) = T(v_1) + T(v_2)}$ $= (-5, -6)$

5. Let $T: V \to V$ be linear, $\boxed{N(T)\ R(T) \text{ are subspaces of } V}$

a) Show that $N(T), R(T)$ are T-invariant.

b) Suppose that $T^2 = T$, show that every $v \in V$ can be written as a sum $v = a + b$ with $a \in N(T), b \in R(T)$, and the expression is unique.

a) ① Let $a \in N(T)$, then $T(a) = \vec{0} \in N(T)$

② let $b \in R(T)$, then $T(b) \in R(T)$

$\cap$

V

T is invariant.

b) $T^2 = T$. $\forall u \in V$

$\Leftrightarrow T(T(u)) = Tu$

b) i) Existence: let $v \in V$.

suppose $v = a + b$, $\boxed{a \in N(T)}$. $b \in R(T)$

$b = T(x)$, $x \in V$

$\Rightarrow v = a + T(x)$. we let T maps both sides.

$\Rightarrow T(v) = T(a) + T^2(x)$

$= T(a) + T(x)$

$= 0 + T(x)$

$\Leftrightarrow T(v) = T(x)$

$\subset T(x) = T(v)$

$\{ \; a = v - T(v).$ $\quad T(a) = T(v) - T^2(v)$

$$= T(v) - T(v) = 0$$

$a \in N(T)$

$$v = \underbrace{(v - T(v))}_{N(T)} + \underbrace{T(v)}_{R(T)}$$

we have showed that There actually exists!

ii) $\boxed{v = (v - T(v)) + T(v).}$

suppose $v = a' + b'$, $\quad a' \in N(T)$

$b' = T(x')$

then $T(v) = T(a') + T(b')$

$$= 0 + T(T(x'))$$

$$= T(x') = b'$$

$T(v) = b' \quad \because \; v = a' + b'$

$\therefore \Rightarrow a' = v - T(v)$

In the end, $\boxed{v = (v - T(v)) + T(v)}$

we proved the uniqueness #

3. Let V be a real vector space, and W_1, W_2 are subspaces of V. The sum of W_1 and W_2 is defined as

$$W_1 + W_2 := \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$$

(a) Show that $W_1 + W_2$ is a subspace of V.

(b)If $W_1 = \operatorname{span}(S_1), W_2 = \operatorname{span}(S_2)$, show that $W_1 + W_2 = \operatorname{span}(S_1 \cup S_2)$

(c) Suppose that $W_1 \cap W_2 = \{0\}$. Then if $R_1 \subset W_1, R_2 \subset W_2$ are linearly independent subsets, show that $R_1 \cup R_2$ is also linearly independent.

proof: Verify three conditions: ① $0_V \in W_1+W_2$

② $\vec{x}+\vec{y} \in W_1+W_2$ for $\forall \vec{x}, \vec{y} \in W_1+W_2$

③ $a\vec{x} \in W_1+W_2$ for $\forall a \in F$ and $\vec{x} \in W_1+W_2$

Firstly, $0_V = 0_V + 0_V \in W_1+W_2$

Secondly, if $\vec{x} = \underset{\in W_1}{w_{11}} + \underset{\in W_2}{w_{12}} \in W_1+W_2$ and

$\vec{y} = \underset{\in W_1}{w_{21}} + \underset{\in W_2}{w_{22}} \in W_1+W_2$

then $\vec{x}+\vec{y} = w_{11}+w_{12} + w_{21}+w_{22}$

$= \underset{\in W_1}{(w_{11}+w_{21})} + \underset{\in W_2}{(w_{12}+w_{22})} \in W_1+W_2$

In the end, let $a \in F(\mathbb{R})$, we have

$a \cdot \vec{x} = a(w_{11}+w_{12}) = \underset{\in W_1}{(aw_{11})} + \underset{\in W_2}{(aw_{12})} \in W_1+W_2$

Thus, we can get W_1+W_2 is a subspace of V.

(b) First, we show $W_1+W_2 \subset span(S_1 \cup S_2)$

For $\forall \vec{x} = w_1+w_2 \in W_1+W_2$, we have,

$\vec{x} = a_1 v_1 + a_2 v_2 + \cdots + a_m v_m$ $\in span(S_1)$

$+ b_1 u_1 + b_2 u_2 + \cdots + b_n u_n$ $\in span(S_2)$

where $v_1 \cdots v_m \in S_1$, $u_1 \cdots u_n \in S_2$

$\therefore \{v_1 \cdots v_m, u_1 \cdots u_n\} \subset S_1 \cup S_2$

$\therefore$ By def, we have $\vec{x} \in span(S_1 \cup S_2)$

Next, for $\forall \vec{x} \in span(S_1 \cup S_2)$, we can have

$\vec{x} = a_1 u_1 + \cdots + a_p u_p$, where $\{u_1 \cdots u_p\} \subset S_1 \cup S_2$

we rearrange all $u_i \in S_1$ and $u_j \in S_2$ and if $u_i \in S_1 \cap S_2$

we put it in S_1. Then we can get

$$\vec{x} = \sum_{u_i \in S_1} a_i u_i + \sum_{u_j \in S_2} a_j u_j \in W_1 + W_2$$

As a result, we have $W_1 + W_2 = span(S_1 \cup S_2)$

(c) Take any elements from R_1 and R_2. For example, $\{u_1 \cdots u_n\} \subset R_1$

$\{v_1 \cdots v_m\} \subset R_2$. Consider:

$$\sum_{i=1}^{n} a_i u_n + \sum_{j=1}^{m} b_j v_j = 0$$

Let $w = \sum_{i=1}^{n} a_i u_n$ $\quad z = \sum_{j=1}^{m} b_j v_j$

① if $w=0$ then $z=0$, then we get

$w = \sum_{i=1}^{n} a_i u_n = 0$ $\quad z = \sum_{j=1}^{m} b_j v_j = 0$

$\because \{u_i\}, \{v_j\}$ are independent respectively

$\therefore \{a_i\}$ and $\{b_j\}$ are all 0.

② if $w = -z \neq 0$, then we have $w = -z \in W_1 \cap W_2$

$(w \in span(\{u_i\}) \subset W_1 \quad -z \in span(\{v_j\}) \subset W_2)$

$\because W_1 \cap W_2 = \{0\}$. $\therefore$ we get $w = -z = 0$. we go back to the first case.

As a result, we have all the elements from $R_1 \cup R_2$ are linearly independent.

4. Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, and k be a positive integer such that $A^k \neq 0$, $A^{k+1} = 0$

(a) Show that $\{I, A, A^2, \ldots, A^k\}$ is linearly independent.

(b) Show that $\{I, A+I, (A+I)^2, \ldots, (A+I)^k\}$ is linearly independent.

proof: (a) consider

$$a_0 I + a_1 A + \cdots + a_k A^k = 0$$

① Multiply $A^k \Rightarrow a_0 A^k + 0 + \cdots + 0 = 0$

$$\Rightarrow a_0 = 0$$

② Multiply A^{k-1} on both sides $\Rightarrow$

$$a_1 A^k + 0 + \cdots + 0 = 0$$

$$\Rightarrow a_1 = 0$$

$\vdots$

repeat similar steps until we get $a_0 \ldots a_{k-1} = 0$. In the end, we only have $a_k A^k = 0 \Rightarrow a_k = 0$

$\therefore$ we have $a_0 \ldots a_k = 0$ for all of them.

(b). First we introduce the binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \quad \text{where } \binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

$$n! = n \times (n-1) \times \cdots \times 2 \times 1$$

Now we consider

$$\sum_{i=0}^{k} a_i (A+I)^i = 0$$

$$\Leftrightarrow \sum_{i=0}^{k} a_i \left(\sum_{j=0}^{i} \binom{i}{j} A^j I^{i-j} \right) = 0$$

$$\Leftrightarrow \sum_{i=0}^{k} \sum_{j=0}^{i} a_i \binom{i}{j} A^j = 0$$

$$\Leftrightarrow \sum_{j=0}^{k} \sum_{i=j}^{k} a_i \binom{i}{j} A^j = 0 \quad \text{(switch } i, j\text{)}$$

$$\Leftrightarrow \sum_{j=0}^{k} \left(\sum_{i=j}^{k} a_i \binom{i}{j} \right) A^j = 0$$

From (a), we know all the coefficients of A^j should be 0.

① when $j = k$

$$\sum_{i=k}^{k} a_i \binom{i}{j} = a_k = 0$$

② when $j = k-1$

$$\sum_{i=k-1}^{k} a_i \binom{i}{j} = a_{k-1} \binom{k-1}{k-1} + a_k \binom{k}{k-1}$$

$$= a_{k-1} + 0 = 0$$

$$\Rightarrow a_{k-1} = 0$$

③ Repeat all these steps, and then we have all $\{a_i\}$ are 0 which completes this proof.