Week 7

MATH 2040

November 4, 2020

1 Problems

1. $\beta = \{(1,1),(1,-1)\}, \beta' = \{(1,0),(1,1)\}$ are basis for $\mathbb{F}^2$, find the change of coordinate matrix that change β' coordinates into β coordinates.

   Ans: $[I]_{\beta'}^{\beta} = [I]_{std}^{\beta}[I]_{\beta'}^{std} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$, where std means the standard basis.

2. Let $T : \mathbb{F}^2 \to \mathbb{F}^2$, $T(a, b) = (2a + b, a - 3b)$, $\beta = \{(1, 1), (1, 2)\}$ is a basis of $\mathbb{F}^2$, find $[T]_\beta$

Ans: $[T]_\beta = [I]_{std}^{\beta}[T]_{std}[I]_{\beta}^{std} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix}$

3. For $A \in M_{n\times n}(\mathbb{F})$, the eigenpolynomial of A is $f_A(t) = \det(A - tI_n)$. Consider linear transformation $T : V \to V$ and β is a basis of V, then the eigenpolynomial of T is $f_T(t) = f_{[T]_\beta}(t)$. Find the eigenvalues of $T(a, b) = (a + 2b, 4a - b)$

   Ans: We know that over standard basis of $\mathbb{F}^2$, $[T]_{std} = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$, then the eigenpolynomial of T is

$$f_T(t) = f_{[T]_std}(t) = \det([T]_{std} - tI) = \begin{vmatrix} 1-t & 2 \\ 4 & -1-t \end{vmatrix} = (1-t)(-1-t) - 8 = t^2 - 9.$$

   So the eigen values of T is ± 3.

4. Let $A, B \in M_{n\times n}(\mathbb{R})$. We say A, B are similar over $\mathbb{R}$ if $A = Q^{-1}BQ$ for some invertible $Q \in M_{n\times n}(\mathbb{R})$ and A, B are similar over $\mathbb{C}$ if $A = Q^{-1}BQ$ for some invertible $Q \in M_{n\times n}(\mathbb{C})$. Prove that A, B are similar over $\mathbb{R}$ equals to A, B are similar over $\mathbb{C}$.

Ans:

- $\Rightarrow$: Obvious
- $\Leftarrow$: Suppose A and B are similar over $\mathbb{C}$, there exists some invertible $Q \in M_n(\mathbb{C})$ such that $A = Q^{-1}BQ$, so $QA = BQ$. Now we rewrite $Q = Q_1 + iQ_2$, where $Q_1, Q_2 \in M_n(\mathbb{R})$, then $QA = BQ$ gives that $Q_1A + iQ_2A = BQ_1 + iBQ_2$, then we have $Q_1A = BQ_1$ and $Q_2A = BQ_2$. Let $f(t) = \det(Q_1 + tQ_2)$, and such f is a degree n polynomial. Since $f(i) = \det(Q_1 + iQ_2) = \det Q \neq 0$, we know that $f \neq 0$, which means it has at most n roots. Then we can find some $t \in \mathbb{R}$ such that $f(t) = \det(Q_1 + tQ_2) \neq 0$. Remark that $\hat{Q} = Q_1 + tQ_2$, then $\hat{Q} \in M_n(\mathbb{R})$ is invertible and

$$\hat{Q}A = Q_1A + tQ_2A = BQ_1 + tBQ_2 = B\hat{Q} \Leftrightarrow A = \hat{Q}^{-1}B\hat{Q},$$

  so A and B are similar over $\mathbb{R}$.

5. Let $A \in M_{n\times n}(\mathbb{F})$, $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct elements of $\mathbb{F}$, $p(t) = (t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_k) \in P_k(\mathbb{F})$. Suppose $p(A) = 0$, where $f(M) = \sum_{i=0}^{k} a_i M^i$ for some $f(t) = \sum_{i=0}^{k} a_i t^i \in P_k(\mathbb{F})$ and $M \in M_{n\times n}(\mathbb{F})$.

(a) Show that for all i there exists a polynomial $f_i \in P_{k-1}(\mathbb{F})$ such that

$$f_i(\lambda_j) = \begin{cases} 1, \text{if } & i = j \\ 0, \text{if } & i \neq j \end{cases}.$$

(b) Show that

i. $Af_i(A) = \lambda_i f_i(A)$

ii. $f_1(A) + f_2(A) + \cdots + f_k(A) = I_n$

iii. $f_i(A)^2 = f_i(A)$

iv. $f_i(A)f_j(A) = 0$ for $i \neq j$

(c) Show that every $v \in \mathbb{F}^n$ can be written uniquely as $v = v_1 + v_2 + \cdots + v_k$ with $v_i \in E_{\lambda_i} = \{v \in \mathbb{F}^n | Av = \lambda_i v\}$.

Ans:

(a) Define the linear transformation $\Phi : P_{k-1}(\mathbb{F}) \to \mathbb{F}^k$, $\Phi(f) = (f(\lambda_1, \lambda_2, \cdots, f(\lambda_k)))$, for all $f \in N(\Phi)$, $\Phi(f) = 0$ means $f(\lambda_i) = 0$ for all $i = 1, 2, \cdots, k$, so f has at least k roots. However, f is degree $k-1$, which means $f = 0$ and $N(\Phi) = \{0\}$. Then $\text{rank}\Phi = k - 0 = k = \dim \mathbb{F}^k$, so Φ is surjective.

(b) i. Let $g_i(t) = tf_i(t) - \lambda_i f_i(t)$, then we have

$$g_i(\lambda_j) = \begin{cases} \lambda_i f_i(\lambda_i) - \lambda_i f_i(\lambda_i) = 0, i = j \\ \lambda_j f_i(\lambda_j) - \lambda_i f_i(\lambda_j) = \lambda_j \cdot 0 - \lambda_i \cdot 0 = 0, i \neq j \end{cases}$$

so $\lambda_1, \lambda_2, \cdots, \lambda_k$ are all roots of g_i. and $g_i(t) = p(t)h_i(t)$ for some $h_i(t)$. Therefore, $g_i(A) = p(A)h_i(A) = 0 = Af_i(A) - \lambda_i f_i(A)$.

ii. Similarly, let $\hat{f}(t) = f_1(t) + f_2(t) + \cdots + f_k(t) - 1$, it's easy to check that $\hat{f}(\lambda_i) = 0$ for all $i = 1, 2, \cdots, k$, so $\hat{f}(t) = p(t)\hat{h}(t)$ for some $\hat{h}(t)$. Therefore, $\hat{f}(A) = 0 = f_1(A) + f_2(A) + \cdots + f_k(A) - I$

iii. Let $f_i'(t) = f_i^2(t) - f_i(t)$, then we have $f_i'(\lambda_j) = \begin{cases} 1^2 - 1 = 0, i = j \\ 0, i \neq j \end{cases}$, so $f_i'(t) = p(t)h_i'(t)$ and $f_i'(A) = p(A)h_i'(A) = 0 = f_i(A)^2 - f_i(A)$.

iv. Let $\tilde{f_{ij}}(t) = f_i(t)f_j(t)$, then we have $\tilde{f_{ij}}(\lambda_a) = f_i(\lambda_a)f_j(\lambda_a)$, for all a, at least one of $f_i(\lambda_a)$ and $f_j(\lambda_a)$ is 0, so $\tilde{f_{ij}}(\lambda_a) = 0$, $\tilde{f_{ij}}(t) = p(t)\tilde{h_{ij}}(t)$. Therefore, $\tilde{f_{ij}}(A) = p(A)\tilde{h_{ij}}(A) = 0 = f_i(A)f_j(A)$.

(c) Exitension: from (b), $f_1(A) + f_2(A) + \cdots + f_k(A) = I$ and $Af_i(A) = \lambda_i f_i(A)$, so for all $v \in \mathbb{F}^k$,

$$v = Iv = f_1(A)v + f_2(A)v + \cdots + f_k(A)v,$$

with $Af_i(A)v = \lambda_i f_i(A)v$, which means $f_i(A)v \in E_{\lambda_i}$.
Uniqueness: suppose $v = v_1 + v_2 + \cdots + v_k$, for any $i \neq j$,

$$\lambda_i f_i(A)v_j = Af_i(A)v_j = f_i(A)Av_j = f_i(A)\lambda_j v_j = \lambda_j f_i(A)v_j.$$

so $(\lambda_i - \lambda_j)f_i(A)v_j = 0$, which means $f_i(A)v_j = 0$. Hence

$$v_i = Iv_i = f_1(A)v_i + \cdots + f_k(A)v_i = f_i(A)v_i.$$

and

$$v_i = f_i(A)v_i = f_i(A)v_1 + \cdots + f_i(A)v_k = f_i(A)v,$$

so all of these v_i are uniquely.