Lecture 24:

Def: Let T be a linear operator on finite-dim inner product space V over F. If $\|T(\vec{x})\| = \|\vec{x}\|$ $\forall \vec{x} \in V$, then we call T is a unitary linear operator. (resp. orthogonal operator) if $F = \mathbb{C}$ (resp $F = \mathbb{R}$)

Lemma: Let U be a self-adjoint linear operator on a fin-dim inner product space V. If $\langle \vec{x}, U(\vec{x}) \rangle = 0$ $\forall \vec{x} \in V$, then $U = T_0$ = zero transf.

Pf: Choose an orthonormal basis β for V consisting of eigenvectors of U.

If $\vec{x} \in \beta$, then $U(\vec{x}) = \lambda \vec{x}$ for some λ.

$$0 = \langle \vec{x}, U(\vec{x}) \rangle = \langle \vec{x}, \lambda \vec{x} \rangle = \bar{\lambda} \langle \vec{x}, \vec{x} \rangle = \bar{\lambda} \|\vec{x}\|^2$$

$$\Rightarrow \quad \lambda = 0$$

$\therefore \; U(\vec{x}) = 0$ for $\forall \, \vec{x} \in \beta$

$\therefore \; U = T_0$.

<u>Thm:</u> For a linear operator T on a fin-dim inner product space V, the following are equivalent:

(a) $TT^* = T^*T = I$

(b) T preserves the inner product on V, i.e.,

$$\langle T(\vec{x}), T(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle \quad \forall \vec{x}, \vec{y} \in V.$$

(c) $T(\underbrace{\beta}_{\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}}) \overset{\text{def}}{=} \{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is an orthonormal basis for V for any orthonormal basis β for V

(d) $\exists$ an orthonormal basis β for V s.t. $T(\beta)$ is an orthonormal basis for V.

(e) $\|T(\vec{x})\| = \|\vec{x}\|$ for $\forall \vec{x} \in V$

Pf: (a) $\Rightarrow$ (b) : $\langle T(\vec{x}), T(\vec{y})\rangle = \langle \vec{x}, \underbrace{T^*T}_{= I}(\vec{y})\rangle = \langle \vec{x}, \vec{y}\rangle$.

(b) $\Rightarrow$ (c) : Let $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be an orthonormal basis for V.

Then: $\langle T(\vec{v}_i), T(\vec{v}_j)\rangle = \langle \vec{v}_i, \vec{v}_j\rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j. \end{cases}$

$\therefore$ $T(\beta)$ is an orthonormal basis for V.

(c) $\Rightarrow$ (d) : Obvious.

(d) $\Rightarrow$ (e) : Let $\vec{x} \in V$ and $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ = orthonormal basis for V.

$\vec{x} = \sum_{i=1}^{n} a_i \vec{v}_i$ for some $a_1, \ldots, a_n \in F$. $\Rightarrow \|\vec{x}\|^2 = \langle \vec{x}, \vec{x}\rangle = \sum_{i=1}^{n} |a_i|^2$ (Check)

But $T(\beta)$ is orthonormal.

$\therefore \|T(\vec{x})\|^2 = \left\|\sum_{i=1}^{n} a_i\, T(\vec{v}_i)\right\|^2 = \sum_{i=1}^{n} |a_i|^2$. $\therefore \|T(\vec{x})\| = \|\vec{x}\|$ for $\forall \vec{x} \in V$.

$(e) \Rightarrow (a)$: $\forall \vec{x} \in V$, $\langle \vec{x}, \vec{x} \rangle = \|\vec{x}\|^2 = \|T(\vec{x})\|^2 = \langle T(\vec{x}), T(\vec{x}) \rangle$

$$= \langle \vec{x}, T^*T(\vec{x}) \rangle$$

$$\Rightarrow \langle \vec{x}, \underbrace{(I - T^*T)}_{U}(\vec{x}) \rangle = 0 \text{ for all } \vec{x} \in V.$$

$\because$ U is self-adjoint $\therefore$ $U = T_0 \Rightarrow I = T^*T$.

Similarly, we can show $TT^* = I$.

Def: A matrix $A \in M_{n\times n}(\mathbb{R})$ is called orthogonal if:

$$A^T A = AA^T = I.$$

The set of orthogonal real matrices is denoted as $O(n)$

A matrix $A \in M_{n\times n}(\mathbb{C})$ is called unitary if:

$$A^* A = AA^* = I$$

The set of unitary complex matrices is denoted as $U(n)$

Remark:

- T is unitary (or orthogonal) iff. $\exists$ an orthonormal basis β s.t. $[T]_\beta$ is unitary (resp. orthogonal)
- Let $\vec{v}_1, \dots, \vec{v}_n \in F^n$. Then $A \overset{\text{def}}{=} \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \in M_{n\times n}(F)$ is unitary (resp. orthogonal) iff $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{C}^n$ (resp $\mathbb{R}^n$)

Thm: Let $A \in M_{n\times n}(\mathbb{C})$. Then A is normal iff A is unitarily equivalent to a diagonal matrix.

(That is, $\exists\, P \in U(n)$ s.t. P^*AP is diagonal)

Pf: $(\Rightarrow)$ Suppose A is normal. Then: $\exists$ an orthonormal basis $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{C}^n$ s.t. $P^{-1}AP = [L_A]_\beta$ is diagonal

(of eigenvectors)

where $P = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}$

$\because$ P is unitary, $P^*P = PP^* = I \Rightarrow P^{-1} = P^*$.

$(\Leftarrow)$ Obvious. Exercise.

__Thm:__ Let $A \in M_{n \times n}(\mathbb{R})$. Then: A is symmetric iff A is orthogonally equivalent to a diagonal matrix.

That is, $\exists\, P \in O(n)$ s.t. $P^T A P$ is diagonal.

e.g.

Consider $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$. Then $\exists P \in O(3)$ s.t. $P^t A P$ is diagonal.

To find P explicitly, we first compute the eigenvalues of A:

$$f_A(t) = (8-t)(2-t)^2$$

So the eigenvalues are $\lambda = 2$ and $\lambda = 8$

For $\lambda = 8$, $(1,1,1)$ is an eigenvector

For $\lambda = 2$, $\{(-1,1,0), (-1,0,1)\}$ is a basis for the eigenspace E_2 but it is not orthogonal.

Applying the Gram-Schmidt process produces the orthogonal basis $\{(-1,1,0), (1,1,-2)\}$ of E_2.

Then an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A is given by

$$\left\{ \frac{1}{\sqrt{2}}(-1,1,0), \frac{1}{\sqrt{6}}(1,1,-2), \frac{1}{\sqrt{3}}(1,1,1) \right\}$$

which gives P as

$$P = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2\sqrt{6} & 1/\sqrt{3} \end{pmatrix}.$$

Goal: $T = \lambda_1 T_1 + \lambda_2 T_2 + \dots + \lambda_k T_k$

Simpler Linear Operators.

Spectral Decomposition

Prop: Let V be an inner product space and $W \subset V$ a fin-dim subspace with an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_k\}$. Then: the orthogonal projection $T : V \to V$ defined by:

$$T(\vec{y}) = \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i$$

is a linear operator s.t.

(1) $N(T) = W^{\perp}$ and $R(T) = W$

(2) $T^2 = T$

(3) T is self-adjoint.

Pf: T is linear because $\langle \cdot, \cdot \rangle$ is linear in the first argument.

$$N(T) = \{\vec{y} \in V : \sum_{i=1}^{k} \langle \vec{y}, \vec{v}_i \rangle \vec{v}_i = \vec{0}\}$$

$$= \{\vec{y} \in V : \langle \vec{y}, \vec{v}_i \rangle = 0 \text{ for } i = 1, 2, \ldots, k\} = W^{\perp}$$

By definition, $R(T) \subset W$.

For $\forall \vec{u} \in W$, we have: $\vec{u} = \sum_{i=1}^{k} \langle \vec{u}, \vec{v}_i \rangle \vec{v}_i = T(\vec{u}) \in R(T)$.

$\therefore \quad W = R(T)$ and $T|_W = I_W$

$$\therefore \quad T^2 = T \circ T = T|_{\underset{W}{\overset{\shortparallel}{R(T)}}} \circ T = I_W \circ T = T$$

For any $\vec{x}, \vec{y} \in V$, write $\vec{x} = \vec{x}_1 + \vec{x}_2 \quad \vec{x}_1 \in W, \quad \vec{x}_2 \in W^{\perp}$

$$\vec{y} = \vec{y}_1 + \vec{y}_2 \qquad \vec{y}_1 \in W, \quad \vec{y}_2 \in W^{\perp}.$$

Then:
$$\begin{array}{ll}
\langle \vec{x}, T(\vec{y}) \rangle & = \langle \vec{x}_1 + \vec{x}_2, \underset{\substack{\| \\ \vec{y}_1}}{T(\vec{y}_1)} + \underset{\substack{\| \\ \vec{0}}}{T(\vec{y}_2)} \rangle = \langle \vec{x}_1, \vec{y}_1 \rangle \\
\quad \color{red}{\|} & \\
\langle T(\vec{x}), \vec{y} \rangle & = \langle \underset{\substack{\| \\ \vec{x}_1}}{T(\vec{x}_1)} + \underset{\substack{\| \\ \vec{0}}}{T(\vec{x}_2)}, \vec{y}_1 + \vec{y}_2 \rangle = \langle \vec{x}_1, \vec{y}_1 \rangle
\end{array}$$

for $\forall \vec{x}, \vec{y} \in V$

$\therefore \quad T^* = T \quad \Rightarrow \quad T$ is self-adjoint.

Thm: Let T be a linear operator on a fin-dim inner product space V over F with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. (spectrum of T)

Assume T is normal (resp. self-adjoint) if $F = \mathbb{C}$ (resp. $F = \mathbb{R}$).

For $i = 1, 2, \ldots, k$, let $E_i = E_{\lambda_i} = \{ \vec{x} \in V : T(\vec{x}) = \lambda_i \vec{x} \}$ and let T_i be the orthogonal projection onto E_i. Then:

(a) $V = E_1 \oplus E_2 \oplus \cdots \oplus E_k$

(b) $E_i^{\perp} = \bigoplus_{j \neq i} E_j$ for $i = 1, 2, \ldots, k$

(c) $T_i T_j = \delta_{ij} T_j$ for $1 \leq i, j \leq k$

(d) $I = T_1 + T_2 + \cdots + T_k$ ← Resolution of the identity operator

(e) $T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$ ← Spectral decomposition

Remark: $V = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ means:

① $V = E_1 + E_2 + \cdots + E_k \overset{\text{def}}{=} \{\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_k : \vec{x}_j \in E_j \text{ for } j = 1, 2, \ldots, k\}$

② $E_i \cap \left(\sum_{j \neq i} E_j\right) = \{\vec{0}\}$ for $\forall\, i \neq j$.

Consequence: ① $\dim(V) = \dim(E_1) + \dim(E_2) + \cdots + \dim(E_k)$

② For any $\vec{v} \in V$,

$\vec{v}$ can be written uniquely as

$$\vec{v} = \underset{\substack{\Uparrow \\ E_1}}{\vec{x}_1} + \underset{\substack{\Uparrow \\ E_2}}{\vec{x}_2} + \cdots + \underset{\substack{\Uparrow \\ E_k}}{\vec{x}_k}$$

Pf: (a) This follows from the fact that T is diagonalizable.

(b) $\because$ $E_j \subset \underset{\overset{\shortparallel}{N(T_i)}}{E_i^{\perp}}$ for $j \neq i$. $\therefore$ $\bigoplus_{j \neq i} E_j \subset E_i^{\perp}$

Now, $\dim(E_i^{\perp}) = \dim(V) - \dim(E_i)$

$= \sum_{j \neq i} \dim(E_j) = \dim\left(\bigoplus_{j \neq i} E_j\right)$

$\therefore$ $E_i^{\perp} = \bigoplus_{j \neq i} E_j$

(c) $T_i T_j = T_i\big|_{\underset{E_j}{\underset{\shortparallel}{R(T_j)}}} T_j = \delta_{ij} I\big|_{E_j} T_j = \delta_{ij} T_j$.

(d) + (e) : $\because$ $V = E_1 \oplus E_2 \oplus \dots \oplus E_k$

$\therefore$ for any $\vec{x} \in V$, $\vec{x}$ can be uniquely written as:

$$\vec{x} = \vec{x}_1 + \vec{x}_2 + \dots + \vec{x}_k \quad , \quad \vec{x}_i \in E_i \quad \text{for} \quad \forall\, i = 1, 2, \dots, k.$$

Then: $T_i(\vec{x}) = \vec{x}_i \Rightarrow (T_1 + T_2 + \dots + T_k)(\vec{x}) = \vec{x}_1 + \vec{x}_2 + \dots + \vec{x}_k$

$$= \vec{x}$$

$$= I(\vec{x})$$

$\therefore$ $T_1 + T_2 + \dots + T_k = I$

Also, $T(\vec{x}) = T(\vec{x}_1) + T(\vec{x}_2) + \dots + T(\vec{x}_k)$

$$= \lambda_1 \vec{x}_1 + \lambda_2 \vec{x}_2 + \dots + \lambda_k \vec{x}_k$$

$$= (\lambda_1 T_1 + \lambda_2 T_2 + \dots + \lambda_k T_k)(\vec{x})$$

$\therefore$ $T = \lambda_1 T_1 + \lambda_2 T_2 + \dots + \lambda_k T_k.$

Cor: If $F = \mathbb{C}$, then T is normal iff $T^* = g(T)$ for some polynomial g.

Pf: $(\Rightarrow)$ Suppose T is normal. Let $T = \lambda_1 T_1 + \lambda_2 T_2 + \dots + \lambda_k T_k$ be spectral decomposition of T.

Then: $T^* = \overline{\lambda_1} T_1^* + \overline{\lambda_2} T_2^* + \dots + \overline{\lambda_k} T_k^*$

$= \overline{\lambda_1} T_1 + \overline{\lambda_2} T_2 + \dots + \overline{\lambda_k} T_k$.

By Lagrange interpolation, $\exists$ a polynomial g s.t. $g(\lambda_i) = \overline{\lambda_i}$ for $i = 1, 2, \dots, k$.

Then: $g(T) = g(\lambda_1 T_1 + \dots + \lambda_k T_k)$

$= g(\lambda_1) T_1 + g(\lambda_2) T_2 + \dots + g(\lambda_k) T_k$

$= \overline{\lambda_1} T_1 + \overline{\lambda_2} T_2 + \dots + \overline{\lambda_k} T_k = T^*$.

$(\Leftarrow)$ If $T^* = g(T)$, then $T^*T = g(T)\,T = T\,g(T)$

$= T\,T^*$.

$\therefore$ T is normal.