Lecture 11:

Recall:

- $\overset{\text{(linear)}}{T} : \underset{\beta}{V} \to \underset{\gamma}{W}$ is invertible iff T is bijective

  $\iff \exists\, T^{-1} \underset{\text{s.t.}}{\ni}$ $T^{-1} \circ T = I_V$ and $T \circ T^{-1} = I_W$

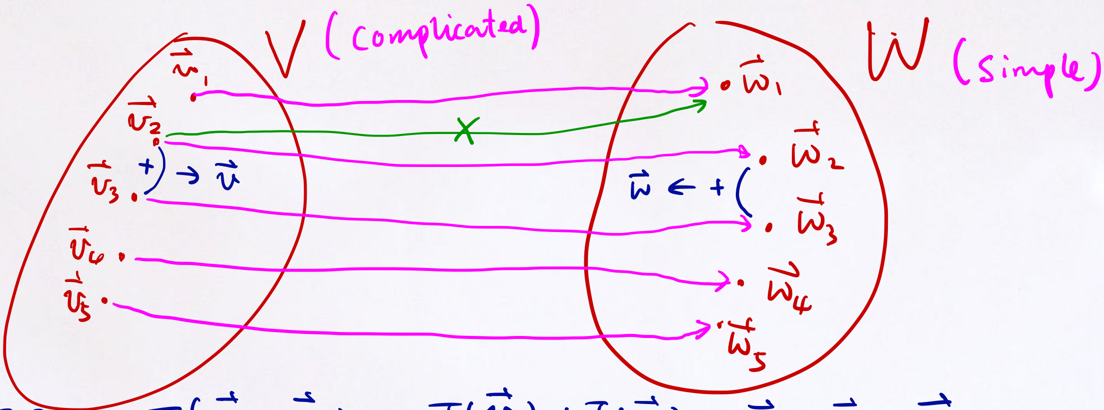
- $T^{-1} : W \to V$ is linear.

- $[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$

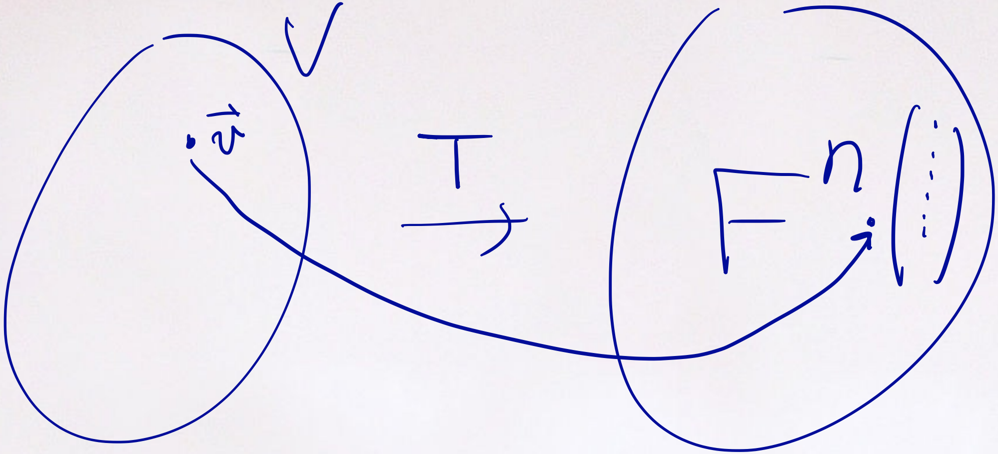
Definition: Let V and W be two vector spaces.

We say V is **isomorphic** to W if $\exists$ an invertible linear transformation $T: V \to W$.

In this case, T is called an **isomorphism** from V onto W.



$$T(\overset{\vec v_2+\vec v_3}{\overset{\parallel}{\vec v}}) = \vec w \text{ ??} \qquad T(\vec v_2+\vec v_3) = T(\vec v_2) + T(\vec v_3) = \vec w_2 + \vec w_3 = \vec w$$


V
$\vec{v}$
T
F^n

Thm: Let V and W be finite-dimensional vector spaces.

Then: V is isomorphic to W iff $\dim(V) = \dim(W)$.

Proof: $(\Rightarrow)$ This direction follows from previous Lemma.

$(\Leftarrow)$: Suppose $\dim(V) = \dim(W) \overset{\text{def}}{=} n$ and let

$\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be basis for V;

$\gamma = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ be basis for W.

Then $\exists$ linear $T: V \to W$ such that $T(\vec{v}_i) = \vec{w}_i$

for $i = 1, 2, \ldots, n$.

By construction, T is onto and $\dim(V) = \dim(W)$.

So, T is one-to-one. $\therefore$ T is invertible.

Corollary: Let V be a vector space over F.
Then: V is isomorphic to F^n iff $\dim(V) = n$

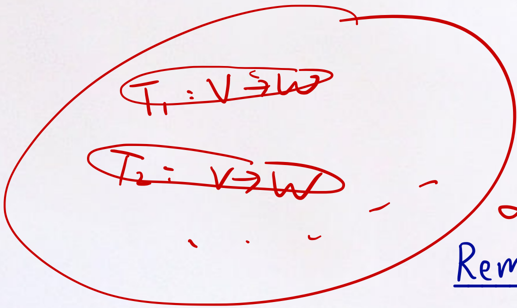
Space of linear transformation

Prop: Let V and W be vector spaces over F.

Then: the set $\mathcal{L}(V,W)$ of all linear transformations from V to W is a vector space over F under the following operations: for linear $T, U : V \to W$, we define: $(T+U) : V \to W$ by $(T+U)(\underset{V}{\vec{x}}) = T(\vec{x}) + U(\vec{x})$

and for any $a \in F$, we define $aT : V \to W$ by

$$(aT)(\underset{V}{\vec{x}}) = a\,T(\vec{x})$$



$\mathcal{L}(V,W)$

Pf: Exercise.

Remark: If $W = V$, we write: $\mathcal{L}(V)$ instead of $\mathcal{L}(V,V)$.

Lemma: Let V and W be finite-dim vector spaces with ordered bases β and γ respectively. Let $T, U : V \to W$ be linear.

Then: (a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

(b) $[aT]_\beta^\gamma = a[T]_\beta^\gamma \quad \forall a \in F$

$$[aT]_\beta^\gamma = \left(\cdots \; [aT(\vec{v}_j)]_\gamma \; \cdots \right)$$

$\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$

Pf: Exercise.

Thm: Let V and W be finite-dimensional vector spaces over F. with dimension n and m respectively. Let β and γ be the ordered bases for V and W respectively.

Then: the map $\Phi : \mathcal{L}(V, W) \longrightarrow M_{m\times n}(F)$ defined by $\Phi(T) = [T]_\beta^\gamma$ is an isomorphism.

Cor: $\dim(\mathcal{L}(V, W)) = \dim(V)\dim(W) = nm$.

Proof: Φ is linear : $\Phi(T+U) = [T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

$= \Phi(T) + \Phi(U)$

$\Phi(aT) = [aT]_\beta^\gamma = a[T]_\beta^\gamma$

$= a\,\Phi(T)$.

Φ is bijective :

For any $A = (A_{ij}) \in M_{m\times n}(F)$, ~~want to show that~~

$\exists!\, T: V \to W$ such that $\cancel{\Phi(T) = [T]_\beta^\gamma} = A$.

$T(\vec{v}_j) = \sum_{i=1}^{m} A_{ij}\vec{w}_i$ for $j = 1, 2, \ldots, m$

$\begin{pmatrix} | \\ [T(\vec{v}_j)]_\gamma \\ | \end{pmatrix}$

$\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, $\gamma = \{\vec{w}_1, \ldots, \vec{w}_m\}$

$\Downarrow$

$\therefore$ For any $A \in M_{m\times n}(F)$, $\exists!\, T: V \to W$ such that $\Phi(T) = A$. (Onto)

(1-1)

$\therefore$ Φ is bijective.

Def Let β be the ordered basis for an n-dimensional vector space V over F. The map $\phi_\beta : V \to F^n$, $\vec{x} \mapsto [x]_\beta$ is called standard representation of V with respect to β.

Prop: ϕ_β is an isomorphism, Given vector spaces V and W of dimension n and m, with ordered bases β and γ respectively. Then, for any $T: V \to W$ (linear), we have:

$$\begin{tikzcd}
\vec{v} \in V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \ni T(\vec{v}) := \vec{w} \arrow[d, "\phi_\gamma"] \\
[\vec{v}]_\beta \in F^n \arrow[r, "L_A"] & F^m \quad [\vec{w}]_\gamma = [T(\vec{v})]_\gamma
\end{tikzcd}$$

where $A = [T]_\beta^\gamma$

$$\Rightarrow \phi_\gamma \circ T(\vec{v}) = L_A \circ \phi_\beta(\vec{v})$$

$$\Leftrightarrow [T(\vec{v})]_\gamma = [T]_\beta^\gamma [\vec{v}]_\beta$$

Change of coordinates

Prop: Let β and β' be two ordered bases for a finite-dim. vector space V, and let $Q = [I_V]_{\beta'}^{\beta}$.

$$V_{\beta'} \xrightarrow{I_V} V_{\beta}$$

Then: (a) Q is invertible

(b) For all $\vec{v} \in V$, $[\vec{v}]_{\beta} = Q[\vec{v}]_{\beta'}$

Proof: (a) Since I_V is invertible, Q is invertible.

(b) Let $\vec{v} \in V$. Then: $[\vec{v}]_{\beta} = [I_V(\vec{v})]_{\beta} = \underbrace{[I_V]_{\beta'}^{\beta}}_{= Q} [\vec{v}]_{\beta'}$

Def: The matrix $Q = [I_V]_{\beta'}^{\beta}$ is called the change of coordinate matrix from β' to β.

Remark: To compute $Q = [I_V]_{\beta'}^{\beta}$, if $\beta = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ and $\beta' = \{\vec{x}_1{}', \vec{x}_2{}', \ldots, \vec{x}_n{}'\}$,

then:

$$Q = \begin{pmatrix} | & & \\ [I_V(\vec{x}_1{}')]_{\beta} & \cdots & \\ | & & \end{pmatrix}$$

$$= \begin{pmatrix} | & & | & \\ [\vec{x}_1{}']_{\beta} & \cdots & [\vec{x}_j{}']_{\beta} & \cdots \\ | & & | & \end{pmatrix}$$

<u>Example:</u> Consider $V = \mathbb{R}^3$.

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \beta' = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Then: $Q = [I_V]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$[\vec{v}]_\beta = Q\,[\vec{v}]_{\beta'}$$

$$\Rightarrow [\vec{v}]_{\beta'} = Q^{-1}[\vec{v}]_\beta$$

$$Q^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Let $\vec{v} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \in \mathbb{R}^3$. Then: $[\vec{v}]_\beta = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

$$[\vec{v}]_{\beta'} = Q^{-1}\,[\vec{v}]_\beta = \begin{pmatrix} 5/2 \\ -1/2 \\ 8/2 \end{pmatrix}$$

Proposition: Let T be a linear operator on finite-dim V. Let β and β' be ordered bases of V. Suppose $Q = [I_V]_{\beta'}^{\beta}$.

Then:
$$[T]_{\beta'} = Q^{-1}[T]_{\beta} Q$$

$$\overset{\beta}{V} \xrightarrow{T} \overset{\beta}{V} \rightsquigarrow [T]_{\beta}$$

$$\overset{\beta'}{V} \xrightarrow{T} \overset{\beta'}{V} \rightsquigarrow [T]_{\beta'}$$

Proof:
$$\begin{aligned} Q[T]_{\beta'} = [I_V]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'} &= [I_V \circ T]_{\beta'}^{\beta} \\ &= [T \circ I_V]_{\beta'}^{\beta} \\ &= [T]_{\beta}^{\beta}[I_V]_{\beta'}^{\beta} \\ &= [T]_{\beta} Q \end{aligned}$$

$$\underset{\beta'}{V} \xrightarrow{I_V} \underset{\beta}{V} \xrightarrow{T} \underset{\beta}{V}$$

Remark: A linear $T: V \to V$ is called linear operator.

Corollary: Let $A \in M_{n\times n}(F)$ and let $\gamma = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$ be an ordered basis for F^n.

Then: $[L_A]_\gamma = Q^{-1} A Q$, $Q = \begin{pmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{pmatrix}$

$\Leftrightarrow [L_A]_\gamma = Q^{-1} [L_A]_\beta Q$

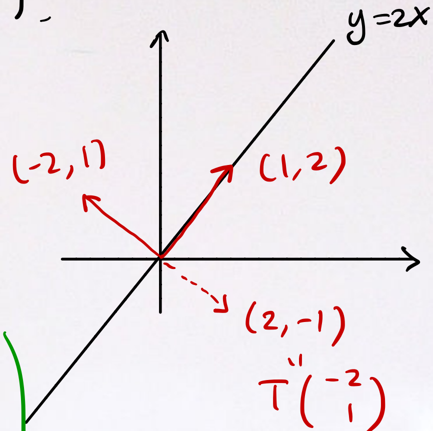
standard ordered basis.

Example: Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the reflection about the line $y=2x$.

Want to compute $[T]_\beta$, where $\beta=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$.

Consider $\beta'=\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}-2\\1\end{pmatrix}\right\}$ for $\mathbb{R}^2$

- $$[T]_{\beta'}=\left(\begin{array}{c|c} \left[T\begin{pmatrix}1\\2\end{pmatrix}\right]_{\beta'} & \left[T\begin{pmatrix}-2\\1\end{pmatrix}\right]_{\beta'}\end{array}\right)$$
$$=\left(\begin{array}{c|c} \left[\begin{pmatrix}1\\2\end{pmatrix}\right]_{\beta'} & \left[\begin{pmatrix}2\\-1\end{pmatrix}\right]_{\beta'}\end{array}\right)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$



- $$Q=[I_{\mathbb{R}^2}]_{\beta'}^{\beta}=\begin{pmatrix}1&-2\\2&1\end{pmatrix}\Rightarrow Q^{-1}=\frac{1}{5}\begin{pmatrix}1&2\\-2&1\end{pmatrix}$$

$$\therefore \quad [T]_{\beta'} = Q^{-1}[T]_\beta Q$$

$$\Longleftrightarrow \quad [T]_\beta = Q \underbrace{[T]_{\beta'}}_{= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} Q^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}.$$

Def: Given two matrices $A, B \in M_{n\times n}(F)$.
We say B is <u>similar</u> to A if $\exists\, Q \in M_{n\times n}$ s.t.

$$B = Q^{-1} A Q.$$