Jordan Form[1][2]

Problem. Let T be a linear operator on a finite-dimensional vector space V. Assume that the characteristic polynomial of T splits; it is always the case for the complex field. Recall that T is diagonalizable if and only if the union of ordered bases for the distinct eigenspaces of T is an ordered base for V. What can we do if T is not diagonalizable, for instance, the dimension of an eigenspace of T associated with an eigenvalue λ is strictly less than the algebraic multiplicity of λ?

Theorem. There exists an ordered base β for V such that

$$[T]_\beta = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}, \tag{1}$$

where the missing entries are all zero and each A_i is a square matrix of the form

$$\text{either } (\lambda) \text{ or } \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}$$

for some eigenvalue λ of T. Such a matrix A_i is called a *Jordan block* corresponding to λ, and the matrix $[T]_\beta$ is called a *Jordan form* of T. We also say that the ordered basis β is a *Jordan basis* for T.

As you can see when reading Chapter 7 of the textbook, the proof of this theorem is not easy. We are not going to repeat to give the full details of the proof in this note. Instead, we assume the existence of the Jordan form and discuss the computations of the Jordan basis and Jordan form.

Definition. A nonzero vector v in V is called a *generalized eigenvector* of T corresponding to λ if $(T-\lambda I)^p(v) = 0$ for some positive integer p. The *generalized eigenspace* of T corresponding to λ, denoted by K_λ, is the subset of V defined by

$$K_\lambda = \{v \in V : (T-\lambda I)^p(v) = 0 \text{ for some postive integer } p\}. \tag{2}$$

[1]This is an additional note for self-learning, and will not be tested in the final exam.

[2]If you have any question to this note, please freely address it to the course instructor Renjun Duan at `rjduan@math.cuhk.edu.hk`.

Let $v \in V$ be a nonzero generalized eigenvector of T corresponding to λ, then there exists a smallest positive integer p such that

$$(T - \lambda I)^p(v) = 0 \ \text{ but } (T - \lambda I)^k(v) \neq 0, \quad k = 1, \cdots, p-1. \tag{3}$$

Then, the ordered set

$$\{(T - \lambda I)^{p-1}(v), (T - \lambda I)^{p-2}(v), \cdots, (T - \lambda I)(v), v\} \tag{4}$$

is linearly independent and called *a cycle of generalized eigenvectors* or *a Jordan chain* of T corresponding to λ and v. In such case, the nonzero vector v is said to be *a generalized eigenvector of rank* p corresponding to λ, so the rank is just the length of the Jordan chain, and v is an ordinary eigenvector if it is of rank $p = 1$. Note that the first vector $(T - \lambda I)^{p-1}(v)$ in the Jordan chain is an ordinary eigenvector of T since it is nonzero and applying $T - \lambda I$ to it gives zero. Also note that the matrix of T corresponding to a specific Jordan chain as above is a Jordan block. This is how we get the matrix representation of T which is block diagonal and where each block is a Jordan block. Moreover, we have the following facts:

- The dimension of the generalized eigenspace K_λ corresponding to an eigenvalue λ is equal to its algebraic multiplicity m_λ defined in terms of the characteristic polynomial of T, that is $\dim K_\lambda = m_\lambda$. Also,

$$K_\lambda = N((T - \lambda I)^{m_\lambda}). \tag{5}$$

- The dimension of each eigenspace E_λ tells us how many cycles of generalized eigenvectors corresponding to λ there are and hence how many Jordan blocks corresponding to λ there are in the Jordan form.

Examples of finding Jordan basis and Jordan form

For a square matrix in each example, find the Jordan form and the Jordan basis.

Example 1. Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 4 & -1 \\ -4 & 13 & -3 \end{pmatrix}. \tag{6}$$

We can view this as either an operator on $\mathbb{C}^3$ or $\mathbb{R}^2$, but it doesn't matter in this case since A has 3 real eigenvalues when counted with multiplicities, so everything we said for operators on complex vector spaces will work even viewing A as an operator on $\mathbb{R}^3$.

The characteristic polynomial of A is

$$f(t) = \det(A - tI) = t(t-1)^2. \tag{7}$$

So, $\lambda_1 = 0$ with $m_1 = 1$ and $\lambda_2 = 1$ with $m_2 = 2$. For the eigenvalue $\lambda_1 = 0$,

$$K_{\lambda_1} = N((A - \lambda_1 I)^{m_1}) = N(A - \lambda_1 I) = E_{\lambda_1}. \tag{8}$$

Hence there is only a single Jordan block of size 1 corresponding to $\lambda_1 = 0$. For the eigenvalue $\lambda_2 = 1$,

$$A - \lambda_2 I = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 3 & -1 \\ -4 & 13 & -4 \end{pmatrix} \to \begin{pmatrix} -1 & 3 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 3 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

Thus, $\dim E_{\lambda_2} = 1$, meaning that there is only one Jordan block corresponding to $\lambda_2 = 1$ in the Jordan form. Since $\lambda_2 = 1$ must appear twice along the diagonal in the Jordan form in terms of the fact that $m_2 = 2$, this single block must be of size 2. Thus the Jordan form of A is

$$\begin{pmatrix} {\color{red}0} & 0 & 0 \\ 0 & {\color{blue}1} & {\color{blue}1} \\ 0 & {\color{blue}0} & {\color{blue}1} \end{pmatrix}, \tag{10}$$

where the colors highlight the two Jordan blocks. The computation of the Jordan basis is left for readers. □

Example 2. Let

$$B = \begin{pmatrix} 5 & -1 & 0 & 0 \\ 9 & -1 & 0 & 0 \\ 0 & 0 & 7 & -2 \\ 0 & 0 & 12 & -3 \end{pmatrix}. \tag{11}$$

This has characteristic polynomial

$$(z-2)^2(z-3)(z-1), \tag{12}$$

so since all eigenvalues are real it again doesn't matter if we consider this to be an operator on $\mathbb{R}^4$ or $\mathbb{C}^4$. From the multiplicities we see that the generalized eigenspaces corresponding to 3 and to 1 are the ordinary eigenspaces, so each of these give blocks of size 1 in the Jordan form.

The eigenspace corresponding to 2 is the null space of

$$B - 2I = \begin{pmatrix} 3 & -1 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 12 & -5 \end{pmatrix}, \tag{13}$$

which row-reduces to

$$\begin{pmatrix} 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{14}$$

This has a 1-dimensional null space, so the eigenspace correspondign to 2 has dimension 1. Thus there is only one Jordan block corresponding to 2 in the Jordan form, so it must of size 2 since 2 has multiplicity 2. Thus, the Jordan form of B is

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

Let's go one step further in this case, and actually find a Jordan basis which puts B into the above form. Recall that this should be a basis consisting of Jordan chains, where each chain corresponds to one Jordan block. For the blocks of size 1, the chains will be of length 1 and will each consist of a single eigenvector for the corresponding eigenvalue. You can check that

$$\begin{pmatrix} 0 \\ 0 \\ -1 \\ 2 \end{pmatrix} \text{ is an eigenvector of } B \text{ with eigenvalue 3, and}$$

$$\begin{pmatrix} 0 \\ 0 \\ -1 \\ 3 \end{pmatrix} \text{ is an eigenvector of } B \text{ with eigenvalue 1.}$$

These give one Jordan chain each. Going back to the row-reduction we did before when finding the dimension of the eigenspace corresponding to 2, we can compute that a basis for the eigenspace corresponding to 2 is given by

$$\begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix}. \tag{16}$$

The final Jordan chain we are looking for (there are only three Jordan chains since there are only three Jordan blocks in the Jordan form of B) much come from this eigenvector, and must be of the form

$$\{(B - 2I)v, v\} \tag{17}$$

since the length has to be the size of the corresponding Jordan block. The first term here should be the ordinary eigenvector we found above, so we must "backtrack" and find a vector v such that

$$(B - 2I)v = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix}. \tag{18}$$

Solving

$$\begin{pmatrix} 3 & -1 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 12 & -5 \end{pmatrix} v = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix} \tag{19}$$

yields

$$v = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \tag{20}$$

as one possible solution. Thus,

$$\{(B - 2I)v, v\} = \{ \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \} \tag{21}$$

is a Jordan chain corresponding to the size 2 Jordan block in the Jordan form of B. Hence, the ordered set

$$\{ \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 3 \end{pmatrix} \} \tag{22}$$

is a Jordan basis corresponding to B, meaning that relative to this basis of $\mathbb{R}^4$ (or $\mathbb{C}^4$) the matrix representation of B is the Jordan form determined before. □

Example 3. Let

$$C = \begin{pmatrix} 1 & -1 & -2 & 3 \\ 0 & 0 & -2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \tag{23}$$

It has the characteristic polynomial

$$(z-1)^4, \tag{24}$$

so we can't immediately say anything about the Jordan form except for the fact that it can only have 1's down the diagonal, since this is the only eigenvalue of C. Next, we determine the dimension of the eigenspace corresponding to 1; by row-reductions,

$$C - I = \begin{pmatrix} 0 & -1 & -2 & 3 \\ 0 & -1 & -2 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & -1 & -2 & 3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{25}$$

which implies that this eigenspace is 2-dimensional. Hence there are two Jordan blocks corresponding to the eigenvalue 1 in the Jordan form. However, this alone does not give us enough information to fully determine the Jordan form since we could have two blocks of size 2 or one block of size 3 and the other of size 1, namely, the Jordan form of C is

$$\text{either} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

To determine which it is, we must determine the lengths of the Jordan chains. We start with ordinary eigenvectors: a basis for the eigenspace corresponding to 1 is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}, \tag{27}$$

which we can get by finding a basis for the null space of $C - I$ using the row-reduction. Each basis vector will give rise a Jordan chain. We start by trying to find v such that

$$(C - I)v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{28}$$

However, when attempting to solve this system of equations you end up with no solution, meaning that there is no such v. Thus, the first eigenvector in (27) cannot appear at the end of a Jordan chain of length greater than 1, so it is its own Jordan chain. For the second eigenvector, we look for a vector w such that

$$(C - I)w = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{29}$$

Solving this system gives

$$w = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \tag{30}$$

as one possible solution. This gives us a Jordan chain of size 2 as

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

To see if there is a Jordan chain of larger length, we next try to find a vector u such that

$$(C - I)u = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}. \tag{31}$$

Solving this gives

$$u = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \tag{32}$$

as a solution, meaning that we have found a Jordan chain of length at least 3 as

$$\{(C - I)^2 u, (C - I)u, u\} = \left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right). \tag{33}$$

We can stop here since we know there is not going to be a Jordan chain of length 4 since the Jordan form does not have a Jordan block of size 4. Thus, since we have found a Jordan chain of length 3 and one of length 1, the Jordan form of C must have a Jordan block of size 3 and one of size 1, so it is of the form in the second case of (26),

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{34}$$

The corresponding Jordan basis is obtained by putting together the Jordan chain of length 3 together with the chain of length 1, namely, the Jordan basis is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}, \tag{35}$$

relative to which the matrix representation of C is the Jordan form given before. □

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