MATH2040A/B Homework 1 Solution

(Sec 1.2 Q08) Ans:

$$\begin{aligned}(a+b)(x+y) &= (a+b)x+(a+b)y \text{ by VS 7} \\ &= ax+bx+ay+by \text{ by VS 8}\end{aligned}$$

(Sec 1.2 Q11) Ans:

(VS 1) Let x, $y \in Z$, $x+y=0+0=y+x$.

(VS 2) For $x,y,z \in Z$,

$$\begin{aligned}(x+y)+z &= (0+0)+0 \\ &= 0+0 \\ &= 0+(0+0) \\ &= x+(y+z)\end{aligned}$$

(VS 3) There exists zero element $0 \in V$, so $x+0=0+0=0=x$

(VS 4) For any $x \in V$, there exists $y=x$ and $x+y=0+0=0=x$

(VS 5) Let $x \in V$, since $c0=0$ for any c, thus $1x=0=x$.

(VS 6) For any $a,b \in F$ and $x \in V$, we have $(ab)x=(ab)0=0=a0=a(bx)$

(VS 7) For any $a \in F$ and $x \in V$, we have $a(x+y)=a(0+0)=a0=0=0+0=a0+a0=ax+ay$

(VS 8) For any $a,b \in F$ and $x \in V$, we have $(a+b)x=(a+b)0=0=0+0=a0+b0=ax+bx$

(Sec 1.2 Q13) Ans: Assume V is a vector space, then the zero element in V is unique. Since $(a_1,a_2)+(0,1)=(a_1+0,a_2)=(a_1,a_2)$,$(0,1)$ is the zero element. For $(1,0)$, there must exists (a,b) such that $(1,0)+(a,b)=(0,1)$, i.e. $(1+a,0*b)=(0,1)$, which is impossible.

(Sec 1.2 Q17) Ans: V isn't a vector space over F. Using (VS8), we have $(a_1,0)=(1+1)(a_1,a_2)=1(a_1,a_2)+1(a_1,a_2)=(a_1,0)+(a_1,0)=(2a_1,0)$, for any $a_1 \in F$. So $a_1=2a_1$ i.e. $a_1=0$ for any $a_1 \in F$. However, F is a field. There is a contradiction.

(Sec 1.2 Q21) Ans: It is obvious that it is closed under multiplication and addition.

(VS 1) Let (s,t), $(x,y) \in Z$, $(s,t)+(x,y)=(s+x,t+y)=(x+s,y+t)=(x,y)+(s,t)$.

(VS 2) For $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in Z$,

$$\begin{aligned}((x_1,y_1)+(x_2,y_2))+(x_3,y_3) &= (x_1+x_2,y_1+y_2)+(x_3,y_3) \\ &= (x_1+x_2+x_3,y_1+y_2+y_3) \\ &= (x_1,y_1)+((x_2+x_3,y_2+y_3)) \\ &= (x_1,y_1)+((x_2,y_2)+(x_3,y_3))\end{aligned}$$

(VS 3) There exists zero element $0_W \in W$, $0_V \in V$, so the zero element in Z is simply $(0_V,0_W)$.

(VS 4) Let $(x,y) \in Z$, then $x \in V$, $y \in W$, there exist $u \in V$, $v \in W$ such that $x+u=0_V$, $y+v=0_W$. Then $(x,y)+(u.v)=(x+u,y+v)=(0_V,0_W)$.

(VS 5) Let $(x,y) \in Z$, $1(x,y)=(1x,1y)=(x,y)$.

(VS 6) Let $a,b \in F$ and $(x,y) \in Z$, $(ab)(x,y)=((ab)x,(ab)y)=(a(bx),a(by))=a(b(x,y))$.

(VS 7) Let $a \in F$ and $(x, y), (u, v) \in Z$, $a((x, y)+(u, v)) = a(x+u, y+v) = (a(x+u), a(y+v)) = (ax + au, ay + av) = (ax, ay) + (au, av) = a(x, y) + a(u, v)$.

(VS 8) Let $a, b \in F$ and $(x, y) \in Z$, $(a + b)(x, y) = ((a + b)x, (a + b)y) = (ax + bx, ay + by) = (ax, ay) + (bx, by) = a(x, y) + b(x, y)$.

(Sec 1.3 Q10) Ans: For W_1, the zero vector belongs to W_1.
For any $x, y \in W_1$, denoting that $x = (a_1, ..., a_n), y = (b_1, ..., b_n)$, we have $x + y = (a_1 + b_1, ..., a_n + b_n)$ since $a_1 + ... + a_n = 0, b_1 + ... + b_n = 0$, we have $(a_1 + b_1) + ... + (a_n + b_n) = (a_1 + ... + a_n) + (b_1 + ... + b_n) = 0$. Therefore $x + y \in W_1$
For any $x \in W_1$ and any $c \in F$, denoting that $x = (a_1, ..., a_n)$, we have $cx = (ca_1, ..., ca_n)$ and $ca_1 + ... + ca_n = c(a_1 + ... + a_n) = 0$. Therefore $cx \in W_1$

For W_2, it is easy to check that $0 \notin W_2$ so that W_2 isn't subspace.

(Sec 1.3 Q19) Ans: Suppose $W_2 \not\subseteq W_1$, there exists $x_2 \in W_2, x_2 \notin W_1$. Let $x_1 \in W_1$, $x_1 + x_2 \in W_1 \cup W_2$, if $x_1 + x_2 \in W_1$, adding $-x_1$ gives $x_2 \in W_1$, contradiction. Therefore $x_1 + x_2 \in W_2$, adding $-x_2$ gives $x_1 \in W_2$.

(Sec 1.3 Q20) Ans: since W is a subspace, we have $a_1w_1, ..., a_nw_n \in W$. Thus $a_1w_1 + a_2w_2 \in W$. Then $a_1w_1 + a_2w_2 + a_3w_3 \in W$, repeating many times, we get the conclusion.

(Sec 1.3 Q23) Ans:

(a) Since W_1, W_2 are subspaces, $0 \in W_1, 0 \in W_2$, $0 = 0+0 \in W_1+W_2$. Let $X = x_1+x_2, y = y_1 + y_2 \in W_1 + W_2$, where $x_1, y_1 \in W_1$, $x_2, y_2 \in W_2$. Then $c(x) = cx_1 + cx_2 \in W_1 + W_2$ since $cx_1 \in W_1$, $cx_2 \in W_2$, and $x+y = (x_1+y_1)+(x_2+y_2) \in W_1+W_2$ since $x_1+y_1 \in W_1$ and $x_2 + y_2 \in W_2$.

(b) Let K be a subspace of V that contains W_1 and W_2, therefore for $x_1 \in W_1$, $x_2 \in W_2$, $x_1 + x_2 \in K$ since K is a subspace and therefore $W_1 + W_2 \subseteq K$.

(Sec 1.3 Q24) Ans: For any $x \in W_1, y \in W_2$, it is obvious that $x + y \in F^n$. So $W_1 + W_2 \subset F^n$. For any $z \in F^n$, denoting $z = (a_1, ..., a_{n-1}, a_n)$, we have $z = ((a_1, ..., a_{n-1}, 0)) + (0, ..., 0, a_n) \in W_1 + W_2$.So $F^n \subset W_1 + W_2$. Thus $F^n = W_1 + W_2$. For any $x = (a_1, ..., a_{n-1}, 0) \in W_1, y = (0, ..., 0, a_n) \in W_2$, if $x = y$, we have $a_i = 0$, for any $i = 1, .., n - 1, n$, i.e. x=y=0. Thus $W_1 \cap W_2 = \{0\}$.