Chapter 2

Change of Variables

Let φ be a continuously differentiable function that maps $[\alpha, \beta]$ into $[a, b]$. For every continuous function f on $[a, b]$, we have following change of variables formula :

$$\int_{\alpha}^{\beta} f(\varphi(y))\varphi'(y)\, dy = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\, dx \ . \tag{2.1}$$

The formula comes from a direct application of the Fundamental Theorem of Calculus. Let $F(x)$ be a primitive function of f, that is, $F' = f$. Consider the composite function $g(y) = F(\varphi(y))$. By the chain rule,

$$g'(y) = F'(\varphi(y))\varphi'(y) = f(\varphi(y))\varphi'(y) \ .$$

By the fundamental theorem of calculus,

$$g(\beta) - g(\alpha) = \int_{\alpha}^{\beta} g'(y)\, dy = \int_{\alpha}^{\beta} f(\varphi(y))\varphi'(y)\, dy \ .$$

On the other hand,

$$g(\beta) - g(\alpha) = F(\varphi(\beta)) - F(\varphi(\alpha)) = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\, dx \ .$$

Hence the formula holds.

When φ maps $[\alpha, \beta]$ bijectively onto $[a, b]$, either φ is strictly increasing with $\varphi(\alpha) = a$, $\varphi(\beta) = b$ or it is strictly decreasing with $\varphi(\alpha) = b$, $\varphi(\beta) = a$. In the first case φ' is non-negative or in the second case non-positive. So (2.1) becomes the formula

$$\int_{\alpha}^{\beta} f(\varphi(y))|\varphi'(y)|\, dy = \int_{a}^{b} f(x)\, dx \ . \tag{2.2}$$

In the first two sections we will extend (1.2) to higher dimension. In the last two sections we consider an extension of (1.1).

2.1 The Change of Variables Formula

Let D_1 and D_2 be two regions in $\mathbb{R}^n$. (Here we are mainly concerned with $n = 2, 3$.) A bijective map from D_1 to D_2 is called a C^1-*diffeomorphism* if it and its inverse are both continuously differentiable.

For a differentiable map Φ from D_1 to $\mathbb{R}^n$, its *Jacobian matrix* $\nabla\Phi$ is given by $(\partial\Phi_i/\partial x_j), i, j = 1, 2, \cdots, n$, that is,

$$\begin{bmatrix} \dfrac{\partial\Phi_1}{\partial x_1} & \cdots & \cdots & \dfrac{\partial\Phi_1}{\partial x_n} \\ \dfrac{\partial\Phi_2}{\partial x_1} & \cdots & \cdots & \dfrac{\partial\Phi_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial\Phi_n}{\partial x_1} & \cdots & \cdots & \dfrac{\partial\Phi_n}{\partial x_n} \end{bmatrix}$$

The determinant of the Jacobian matrix is called the *Jacobian* of Φ. It will be denoted by J_Φ.

By the Inverse Function Theorem, a C^1-map from a region D in $\mathbb{R}^n$ to $\mathbb{R}^n$ which is one-to-one and whose Jacobian never vanishes sets up a C^1-diffeomorphism between D and its image $\Phi(D)$. This fact will be used implicitly and frequently below.

Theorem 2.1. ***(Change of Variables Formula)*** *Let Φ be a C^1-diffeomorphism from D_1 to D. For any continuous function f in D,*

$$\int_D f(\boldsymbol{x})\, d\boldsymbol{x} = \int_{D_1} f(\Phi(\boldsymbol{y}))|J_\Phi(\boldsymbol{y})|\, d\boldsymbol{y}\ . \tag{2.3}$$

Here $d\mathbf{x}$ and $d\mathbf{y}$ refer to the integration over an n-dimensional region. For $n = 2$, in our usual notation, this formula reads as,

$$\iint_D f(x, y)\, dA(x, y) = \iint_{D_1} f(\Phi(u, v))|J_\Phi(u, v)|\, dA(u, v)\ ,$$

and for $n = 3$,

$$\iiint_\Omega f(x, y, z)\, dV(x, y, z) = \iiint_{\Omega_1} f(\Phi(u, v, w))|J_\Phi(u, v, w)|\, dV(u, v, w)\ .$$

The integration formulas for the polar coordinates, cylindrical coordinates and spherical coordinates are special cases of this theorem.

In the case of the polar coordinates, we take $n = 2$ and $\Phi(r, \theta) = (r\cos\theta, r\sin\theta)$. Then $J_\Phi = r \geq 0$, so the formula (2.3) becomes

$$\iint_D f(x, y)\, dA(x, y) = \iint_{D_1} f(r\cos\theta, r\sin\theta) r\, dA(r, \theta) \ .$$

In the case of the cylindrical coordinates, we take $n = 3$ and $\Phi(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$. Then $J_\Phi = r$ and (2.3) becomes

$$\iiint_\Omega f(x, y, z)\, dV = \iint_{\Omega_1} f(r\cos\theta, r\sin\theta, z) r\, dV(r, \theta, z) \ .$$

when

In the case of the spherical coordinates, we take $n = 3$ and

$$\Phi(\rho, \varphi, \theta) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) \ , \quad \varphi \in [0, \pi], \ \theta \in [0, 2\pi) \ .$$

Then $J_\Phi = \rho^2 \sin\varphi \geq 0$ and (2.3) becomes

$$\iiint_\Omega f(x, y, z)\, dV = \iiint_{\Omega_1} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2 \sin\varphi\, dV(\rho, \varphi, \theta) \ .$$

We now explain the ideas behind (2.3).

We take $n = 2$ and D_1 a rectangle. A partition $P = \{R_{ij}\}$ on D_1 introduces a generalized partition $\{D_{ij}\}$ on D. Now, for a continuous function f in D, when the partition P becomes very fine, by Theorem 1.10,

$$\begin{aligned} \iint_D f\, dA &\approx \sum_{i,j} f(p_{ij})|D_{ij}| \\ &= \sum_{i,j} f(\Phi(q_{ij}))\frac{|D_{ij}|}{|R_{ij}|}|R_{ij}| \ , \end{aligned}$$

where p_{ij} is a tag point in D_{ij} and $\Phi(q_{ij}) = p_{ij}$. This is possible because Φ is bijective.

Now, let us focus on a subrectangle R_{ij}. Let $(u, v), (u+h, v), (u, v+k), (u+h, v+k)$ be the vertices of the subrectangle. (We have dropped the subscripts i, j for simplicity. (u, v) should be (u_i, v_j) and $h = \Delta x_i, k = \Delta y_j$.) Its image D_{ij} has vertices at $\Phi(u, v), \Phi(u + h, v), \Phi(u, v + k)$, and $\Phi(u + h, v + k)$. By Taylor's expansion,

$$\Phi(u + h, v) = \Phi(u, v) + \Phi_u(u, v)h + \text{ higher order terms},$$

$$\Phi(u, v+k) = \Phi(u,v) + \Phi_v(u,v)k + \text{ higher order terms},$$

and

$$\Phi(u+h, v+k) = \Phi(u,v) + \Phi_u(u,v)h + \Phi_v(u,v)k + \text{ higher order terms} .$$

Ignoring the higher order terms, D_{ij} is well approximated by the parallelogram with vertices at $\Phi(u,v), \Phi(u,v) + \Phi_u(u,v)h, \Phi(u,v) + \Phi_v(u,v)k,$ and $\Phi(u,v) + \Phi_u(u,v)h + \Phi_v(u,v)k$. Recall that for a parallelogram spanned by two vectors (a_1, a_2) and (b_1, b_2), its area is given by $|a_1b_2 - a_2b_1|$. Therefore, the area of our parallelogram is equal to $|J_\Phi(u,v)|hk$. As hk is just the area of R_{ij}, so

$$\frac{|D_{ij}|}{|R_{ij}|} \approx \frac{|J_\Phi(u_i, v_j)|hk}{hk} = |J_\Phi(u_i, v_j)|.$$

It follows that

$$\sum_{i,j} f(\Phi(q_{ij}))\frac{|D_{ij}|}{|R_{ij}|}|R_{ij}| \approx \sum_{i,j} f(\Phi(q_{ij}))|J_\Phi(u_i, v_j)||R_{ij}| .$$

Note that (u_i, v_j) is also a tag point in R_{ij}. Applying Theorem 1.11, as $\|P\| \to 0$,

$$\iint_D f(x,y)\, dA(x,y) = \iint_{D_1} f(\Phi(u,v))|J_\Phi|(u,v)\, dA(u,v) .$$

Similarly, in $n = 3$, the subrectangular box B_{ijk} maps to a parallelepiped Ω_{ijk} under Φ and the volume ratio

$$\frac{|\Omega_{ijk}|}{|B_{ijk}|} \approx |J_\Phi(u_i, v_j, w_k)| .$$

In the following we look at some examples. We point out that in $n = 2, 3$, people like to use another notation for the Jacobian matrix, for instance, J_Φ is written as

$$\frac{\partial(x,y)}{\partial(u,v)} .$$

The variables in the numerator and denominator are respective the dependent and independent variables. In the next section we will establish the useful relation:

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} .$$

Example 2.1. Find the area of the region bounded by the curves $y = x, y = 6x, xy = 1$ and $xy = 5$.

We make the region simpler by introducing the change of variables $u = y/x$ and $v = xy$. The rectangle $(u, v) \in [1, 6] \times [1, 5]$ is mapped to the region under $\Phi : (u, v) \mapsto (x, y)$. The map Φ can be determined by expressing x, y in terms of u, v. After a little manipulation, we get $x = \sqrt{vu^{-1}}, y = \sqrt{uv}$. The Jacobian is equal to $1/(-2u)$. It follows that the area is given by

$$\begin{aligned}
\iint_D 1\, dxdy &= \int_1^6 \int_1^5 \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv \\
&= \int_1^6 \int_1^5 \left| \frac{1}{-2u} \right| dvdu \\
&= 2 \log 6 \ .
\end{aligned}$$

We point out one can determine the Jacobian without Φ. Indeed, the Jacobian of the inverse map is

$$\frac{\partial(u, v)}{\partial(x, y)} = -2y/x = -2u.$$

By the relation above, the Jacobian of Φ is $1/(-2u)$.

Example 2.2. Evaluate the iterated integral

$$\int_0^1 \int_0^{1-x} \sqrt{x + y}(y - 2x)^2\, dydx \ .$$

This is a double integral over the triangle with vertices at $(0, 0), (1, 0)$ and $(0, 1)$. While the region of integration is simple enough, the integrand is a bit messy. Unlike the first example, we simplify the integrand this time. Letting $u = x + y$ and $v = y - 2x$, the integrand becomes $\sqrt{u}v^2$ but the region becomes the region bounded by the curves $x = 0, y = 0, x + y = 1$ which go over to $u = v, 2u + v = 0$ and $u = 1$. The Jacobian of the inverse map is

$$\frac{\partial(u, v)}{\partial(x, y)} = 3 \ .$$

Therefore,

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \sqrt{x + y}(y - 2x)^2\, dydx &= \int_0^1 \int_{-2u}^{u} \sqrt{u}v^2 \frac{1}{3}\, dydx \\
&= \frac{2}{9} \ .
\end{aligned}$$

Example 2.3 Evaluate

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}} e^{\sqrt{xy}} \, dxdy \ .$$

The region is composed three sides given by $y = x, xy = 1$ and $y = 2$. Let $u = \sqrt{xy}$ and $v = \sqrt{y/x}$ or $x = u/v, y = uv$. The region goes over to the region bounded by $v = 1, u = 1$ and $uv = 2$. We have

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{2u}{v} \ .$$

Therefore, our integral is equal to

$$\int_1^2 \int_1^{2/u} ve^u \frac{2u}{v} \, dudv = 2e(e-2) \ .$$

Next we look at some three dimensional examples.

Example 2.4 Evaluate

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=y/2+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dxdydz \ .$$

The region projected to the rectangle $[0,3] \times [0,4]$ in yz-plane and is simple enough. Let $t = x - y/2 \in [0,1], y = y, z = z$ be the change of variables. The Jacobian is equal to 1. Therefore, this integral is equal to

$$\int_0^3 \int_0^4 \int_0^1 \left(t + \frac{z}{3} \right) dtdydz = 12 \ .$$

Example 2.5. Find the volume of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1$.

Introducing the change of variables $x = au, y = bv, z = cw$, the ellipsoid is the image of the unit ball B, $u^2 + v^2 + w^2 \le 1$. We have

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = abc \ .$$

Therefore, the volume of the ellipsoid is given by

$$\iiint_B 1 \times abc \, dV(u,v,w) = \frac{4}{3}\pi abc \ .$$