Chapter 4

The Divergence Theorem

In this chapter we discuss formulas that connects different integrals. They are

(a) Green's theorem that relates the line integral of a vector field along a plane curve to a certain double integral in the region it encloses.

(b) Stokes' theorem that relates the line integral of a vector field along a space curve to a certain surface integral which is bounded by this curve.

(c) Gauss' theorem that relates the surface integral of a closed surface in space to a triple integral over the region enclosed by this surface.

All these formulas can be unified into a single one called the divergence theorem in terms of differential forms.

4.1 Green's Theorem

Recall that the fundamental theorem of calculus states that

$$\int_a^b f'(x)\,dx = f(b) - f(a)\ .$$

It relates the integral of the derivative of a function over an interval $[a, b]$ to the endpoint values of the function. In higher dimension we replace the function by a vector field. A possible two dimensional extension would be a formula relating the double integral of some quantity involving the partial derivatives of a vector field to the line integral of the vector field along its boundary curve. This is the content of Green's theorem.

Theorem 4.1. ***(Green's Theorem)*** *Let $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$ be a C^1-vector field in an open set G in the plane. Suppose C is a simple, closed curve in G and the set D it bounds lies completely inside G. Then*

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_C M\,dx + N\,dy \ , \tag{4.1}$$

where C is oriented in the anticlockwise way.

A simple, closed curve divides the plane into two regions, one bounded and the other unbounded. Here C bounds D means D is the bounded region.

Recall that line integral

$$\oint_C M\,dx + N\,dy = \oint_C \mathbf{F}\cdot\,d\mathbf{r} \ ,$$

is called the *circulation* of $\mathbf{F}$ around the closed curve C. When $\mathbf{F}$ is the velocity vector field of some fluid, its circulation around a curve measures the amount of the fluid flowing around the curve in unit time. When an admissible parametrization $\mathbf{r} : [a, b] \mapsto C$ is chosen, the line integral can be evaluated by the formula

$$\oint_C M\,dx + N\,dy = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \ .$$

Proof. We will prove Green's theorem in a special case, namely, D can be expressed simultaneously in the following two ways:

$$D = \{(x, y) : \ f_1(x) \le y \le f_2(x), \ x \in [a, b]\}$$

and

$$D = \{(x, y) : \ g_1(y) \le x \le g_2(y), \ y \in [c, d]\} \ .$$

Typical examples of such regions include ellipses and rectangles.

We shall show that

$$\iint_D \frac{\partial M}{\partial y}\,dA = -\oint_C M\,dx \ , \tag{4.2}$$

and

$$\iint_D \frac{\partial N}{\partial x}\,dA = \oint_C N\,dy \ . \tag{4.3}$$

Green's theorem follows by adding (4.2) and (4.3) together.

The boundary curve C of D consists of the four curves:

$$C_1 : \quad \mathbf{r}_1(x) = (x, f_1(x)), \quad x \in [a, b] \ ,$$

$$C_2 : \quad \mathbf{r}_2(x) = (x, f_2(x)), \quad x \in [a, b] \ ,$$

$$\gamma_2: \quad \gamma_2(y) = (b, y), \quad y \in [f_1(b), f_2(b)] \ ,$$
$$\gamma_1: \quad \gamma_1(y) = (a, y), \quad y \in [f_1(a), f_2(b)] \ ,$$

where γ_1 and γ_2 may degenerate into points. We have $C = C_1 + \gamma_2 - C_2 - \gamma_1$.

By Fubini's theorem

$$\begin{aligned}
\iint_D \frac{\partial M}{\partial y}\, dA &= \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}(x, y)\, dydx \\
&= \int_a^b M(x, y)\Big|_{f_1(x)}^{f_2(x)} dx \\
&= \int_a^b M(x, f_2(x)) - M(x, f_1(x))\, dx \\
&= -\int_{C_1 - C_2} M\, dx \ .
\end{aligned}$$

On the other hand, $\gamma_1'(y) = (0, 1)$, so

$$\int_{\gamma_1} M\, dx = \int_{f_1(a)}^{f_2(a)} M(a, y) x'(y)\, dy = 0,$$

as $x'(y) \equiv 0$. By the same reasoning

$$\int_{\gamma_2} M\, dx = 0 \ ,$$

too. Therefore,

$$\iint_D \frac{\partial M}{\partial y}\, dA = -\int_{C_1 - C_2} M\, dx = \oint_{C_1 + \gamma_2 - C_2 - \gamma_1} M\, dx = -\oint_C M\, dx \ ,$$

and (4.1) follows. Similarly, we can prove (4.2).

When the region D is of more complicated geometry, one can use horizontal and vertical lines to decompose it into the union of regions of the above types. We will not go into the details. □

We will discuss four applications of Green's theorem:

- Evaluation of line integrals,
- Study independence of path,
- An area formula,
- Localizing divergence and rotation.

The first application is illustrated in the following example.

Example 4.1. Evaluate

$$\oint_C -y^2\,dx + xy\,dy \ ,$$

where C is the boundary of the square at $(0,0),(1,0),(1,1)$ and $(0,1)$ in anticlockwise direction.

A direct evaluation is not difficult, but tedious as it involves evaluating four line integrals. Instead we take advantage of Green's theorem

$$\begin{aligned}\oint_C -y^2\,dx + xy\,dy &= \iint_R \left(\frac{\partial xy}{\partial x} - \frac{\partial - y^2}{\partial y}\right) \\ &= \iint_R 3y\,dA(x,y) \\ &= \int_0^1\int_0^1 3y\,dxdy \\ &= \frac{3}{2}\ .\end{aligned}$$

Next, we return to the discussion on independence of path of vector fields in Chapter 3. We established Theorem 3.4 which asserts that a vector field in $\mathbb{R}^n$ is conservative if and only if the compatibility condition (3.9) holds (when $n=2$). Now, by using Green's theorem, we will present a more general result.

A region in $\mathbb{R}^n$ is called *simply connected* if it is connected and every closed curve lying in it can be deformed continuously to a point inside the set itself. The entire plane, a disk, a convex set and more general a star-shaped region are examples of simply connected sets in the plane. On the other hand, a punctured disk (a disk with the center removed) and an annulus are examples of connected but not simply-connected regions. Roughly speaking, simply connected regions are those connected regions which do enclose any holes.

Theorem 4.2. *Let $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$ be a C^1-vector field in a simply connected region G in the plane. It is conservative if and only if the compatibility condition holds:*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\ . \tag{4.4}$$

This generalizes Theorem 3.4 where it is required the vector field to be defined in the entire space.

Proof. In Chapter 3, it was shown that (4.4) (that is (3.9)) holds when the vector field $\mathbf{F}$ is conservative. Conversely, under (4.4) a potential function were constructed under the assumption that the line integrals along all simple curves connecting two points have the same value. It suffices to verify this property in the present situation. Let γ_1 and γ_2 be two simple curves connecting point A to point B. When these two curves do not intersect, $\gamma \equiv \gamma_1 - \gamma_2$ forms a simple closed curve. Green's theorem implies that

$$\oint_\gamma M\,dx + N\,dy = 0 \ ,$$

hence

$$\int_{\gamma_1} M\,dx + N\,dy = \int_{\gamma_2} M\,dx + N, dy \ .$$

When γ_1 and γ_2 intersect, we may add another curve γ_3 connecting A and B so that it does not intersect γ_1 and γ_2. Thus $\gamma_1 - \gamma_3$ and $\gamma_2 - \gamma_3$ form simple closed curves respectively. Using

$$\int_{\gamma_1} M\,dx + N\,dy = \int_{\gamma_3} M\,dx + N\,dy = \int_{\gamma_2} M\,dx + N\,dy \ ,$$

we draw the same conclusion. Using this property one can define the potential of $\mathbf{F}$ as in the proof of Theorem 3.4. The existence of a potential Φ shows that

$$\int_A^B \mathbf{F}\cdot\mathbf{t}\,ds = \Phi(B) - \Phi(A) \ ,$$

along any path from A to B in G no matter the path is simple or not. We conclude that $\mathbf{F}$ is conservative. □

Green's theorem is a formula relating the line integral of a curve to a double integral of the region it encloses. This observation leads to a formula expressing the area of the region in terms of a boundary integral.

Let A be the area of the region enclosed by a simple closed curve C in the plane. Applying Green's theorem to the vector field $y\mathbf{i}$ yields

$$A = -\oint_C y\,dx \ . \tag{4.5}$$

Similarly, choosing the vector field to be $x\mathbf{j}$ yields

$$A = \oint_C x\,dy \ . \tag{4.6}$$

These two formulas together implies a more symmetric formula for the area:

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy \ . \tag{4.7}$$

These formulas express the area enclosed by a curve in terms of the curve. It has interesting geometric consequence. For instance, together with Fourier series, the last formula can be used to prove the classical isoperimetric inequality, that is, among all regions enclosed by a simple closed curve with the same perimeter, only the disk has the largest area.

Finally, recall that in Chapter 3 we introduce the concept of the circulation and the flux of a vector along a curve. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be C^1-vector field defined on the simple closed oriented curve C with the chosen tangent $\mathbf{t}$ and normal $\mathbf{n}$. The circulation and the flux of $\mathbf{F}$ around C is defined to

$$\oint_C M\,dx + N\,dy\ ,$$

and

$$\oint_C M\,dy - N\,dx\ ,$$

respectively. Green's theorem suggests a way to define the circulation and the flux of a vector field at a point. In other words, we can localize circulation and flux.

Let $\mathbf{p}$ be a point in some open set $G \subset \mathbb{R}^2$ where a C^1-vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is defined. Let C be a simple, closed curve anitclockwisely oriented enclosing $\mathbf{p}$ in its interior, and D the region it bounds. The quantity

$$\begin{aligned}\frac{1}{|D|}\oint_C M\,dx + N\,dy &= \frac{1}{|D|}\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \\ &\to \frac{\partial N}{\partial x}(\mathbf{p}) - \frac{\partial M}{\partial y}(\mathbf{p})\ .\end{aligned}$$

In view of this, we define the *curl* (or the *rotation*) of $\mathbf{F}$ at $\mathbf{p}$ to be

$$\text{rot }\mathbf{F}(\mathbf{p}) = \left(\frac{\partial N}{\partial x} - \frac{\partial Q}{\partial y}\right)(\mathbf{p})\ .$$

Similarly, the flux of $\mathbf{F}$ across C is equal to

$$\oint_C M\,dy - N\,dx\ .$$

By Green's theorem,

$$\begin{aligned}\frac{1}{|D|}\oint_C M\,dy - N\,dx &= \frac{1}{|D|}\iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA \\ &\to \frac{\partial M}{\partial x}(\mathbf{p}) + \frac{\partial N}{\partial y}(\mathbf{p})\ .\end{aligned}$$

Hence, we define the *divergence* (or *flux density*) of $\mathbf{F}$ at $\mathbf{p}$ to be

$$\text{div }\mathbf{F}(\mathbf{p}) = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)(\mathbf{p})\ .$$

Example 4.2. Evaluate

$$\oint_C \frac{y}{x^2+y^2}\,dx + \frac{-x}{x^2+y^2}\,dy \ ,$$

where C is the ellipse $x^2/4 + y^2/9 = 1$ oriented in positive direction.

By a direct computation, the vector field

$$\frac{y}{x^2+y^2}\mathbf{i} + \frac{-x}{x^2+y^2}\mathbf{j}$$

satisfies $M_y = N_x$. See the end of Section 3.6 in Chapter 3. However, since it is not defined at the origin, we cannot appeal to Green's theorem to conclude that this line integral vanishes. What we could do is to change it to an easier line integral. In fact, let C_r be a small circle centered at the origin and is contained inside C. We orient C_r in clockwise direction and connect C_r to C by the line segment L which runs from $(r,0)$ to $(2,0)$. Then $\Gamma = C - L + C_r + L$ forms a closed curve enclosing a simply-connected domain. Γ is not simple, but we can lift $\pm L$ up a little bit to make it simple. Applying Green's Theorem to this simple, closed curve and then passing to limit, we have

$$\begin{aligned} 0 &= \oint_\Gamma M\,dx + N\,dy \\ &= \left(\int_C - \int_L + \int_{C_r} + \int_L\right) M\,dx + N\,dy \\ &= \left(\int_C + \int_{C_r}\right) M\,dx + N\,dy \ . \end{aligned}$$

Therefore,

$$\begin{aligned} \int_C \frac{y}{x^2+y^2}\,dx + \frac{-x}{x^2+y^2}\,dy &= -\int_{C_r} \frac{y}{x^2+y^2}\,dx + \frac{-x}{x^2+y^2}\,dy \\ &= \int_0^{2\pi} (\cos\theta\cos\theta + (-\sin\theta)(-\sin\theta))\,dt \\ &= 2\pi \ . \end{aligned}$$

The trick of adding an artificial line segment to form a simply conneced region in this example leads us to the general form of Green's Theorem. Let D be a region bounded by several simple, closed curves $C_1, C_2, \cdots, C_n$ where C_1 is the outer one and the rest are sitting inside C_1.

Theorem 4.3. *Let $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j}$ be a C^1-vector field in D. Then*

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \sum_{j=1}^n \oint_{C_j} P\,dx + Q\,dy \ ,$$

where C_1 is oriented in anticlockwise way and $C_j, j \geq 2$, are in clockwise way.