Tutorial 3 (3 Feb)

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Fubini's Theorem in polar coordinates

Thm (Fubini's Theorem for continuous functions in polar coordinates)

Let $f: D \to \mathbb{R}$ be a continuous function over a region D of the form

$D = \{ (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \mid \theta_1 \leq \theta \leq \theta_2 ; \ \varphi_1(\theta) \leq r \leq \varphi_2(\theta) \}$, where

- $\theta_1, \theta_2 \in [0, 2\pi)$ (or $[-\pi, \pi)$, or any $[a, a+2\pi)$, $a \in \mathbb{R}$) are constants satisfying $\theta_1 < \theta_2$.
- $\varphi_1, \varphi_2 : [\theta_1, \theta_2] \longrightarrow \mathbb{R}$ are continuous satisfying $0 \leq \varphi_1(\theta) \leq \varphi_2(\theta)$ for any $\theta \in [\theta_1, \theta_2]$.

then $\iint_D f \, dA = \int_{\theta_1}^{\theta_2} \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(r\cos\theta, r\sin\theta) \, r \, dr \, d\theta$.

Cor (Area of a region via a double integral in polar coordinates)

Given a region D as above, then its area is

$$\text{Area}(D) \overset{\text{Def}}{:=} \iint_D 1 \cdot dA \overset{\text{Thm}}{=} \int_{\theta_1}^{\theta_2} \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} r \, dr \, d\theta$$

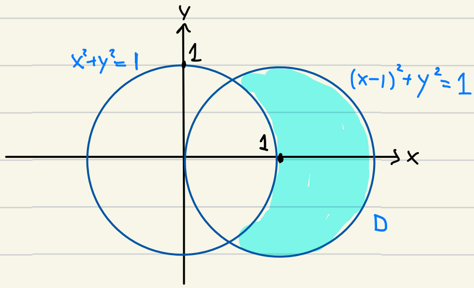
$$= \frac{1}{2} \int_{\theta_1}^{\theta_2} \left((\varphi_2(\theta))^2 - (\varphi_1(\theta))^2 \right) d\theta$$

Ex Find the area of the region inside the circle $(x-1)^2+y^2=1$

and outside the circle $x^2+y^2=1$.

Sol Idea: Determine the polar coordinate description of the region.

Step 1: Sketch the region D.



Step 2: Describe D in terms of polar coordinates.

Put $x=r\cos\theta,\ y=r\sin\theta$: $\begin{cases} x^2+y^2=1 \iff r=1 \\ (x-1)^2+y^2=1 \iff (x^2+y^2)=2x \iff r=2\cos\theta. \end{cases}$

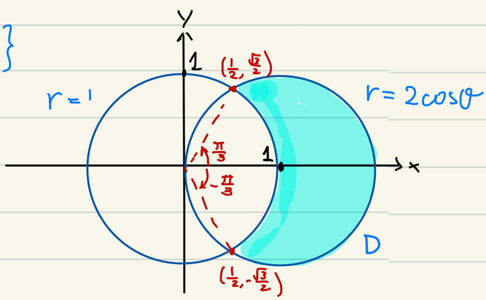
(where $r>0,\ -\pi\leq\theta<\pi$)

$\therefore$ Intersection points of two circles satisfy $r=2\cos\theta=1$.

$\therefore$ $\theta=\frac{\pi}{3}$ or $-\frac{\pi}{3}$, and $r=1$.

Hence, the coordinates of intersection points are $(x,y)=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ or $\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$.

$D=\left\{(r,\theta)\in(0,+\infty)\times[-\pi,\pi)\ \middle|\ -\frac{\pi}{3}\leq\theta\leq\frac{\pi}{3},\ 1\leq r\leq 2\cos\theta\right\}$



Step 3: Evaluate the area.

$$\text{Area}(D) = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{1}^{2\cos\theta} r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[\frac{r^2}{2}\right]_{1}^{2\cos\theta} d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (4\cos^2\theta - 1)\,d\theta$$

$$= \int_{0}^{\frac{\pi}{3}} (4\cos^2\theta - 1)\,d\theta$$

$$= \int_{0}^{\frac{\pi}{3}} \left(4\cdot\left(\frac{1+\cos 2\theta}{2}\right) - 1\right) d\theta$$

$$= \int_{0}^{\frac{\pi}{3}} (1 + 2\cos 2\theta)\,d\theta$$

$$= \left[\theta + \sin 2\theta\right]_{0}^{\frac{\pi}{3}} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}.$$