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Office Hour: Send me an email first, then we will arrange a meeting (if you need it).
Tutorial Arrangement:

- (1330 - 1355/ 15:30 - 15:55): Problems.
- (1355 - 1415/ 15:55 - 16:15): Class exercises.
- (1415 - 1430/ 16:15 - 16:30): Submission of Class Exercise via Gradescope.
- (1430 - 1530/ 16:30 - 17:30): Late submission period.

1 More Green's Theorem

Green's theorem can be extended to regions with holes, that is, regions that are not simply-connected.

$$\begin{aligned}\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA &= \iint_{D'} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA + \iint_{D''} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \\ &= \int_{\partial D'} M\,dx + N\,dy + \int_{\partial D''} M\,dx + N\,dy\end{aligned}$$

Since the integral alone the common boundary is cancelled out, hence

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy = \int_C M\,dx + N\,dy.$$

Exercise

Q1

If $\mathbf{F}(x,y) = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2+y^2}$, show that $\int_C \mathbf{F}\cdot d\mathbf{r} = 2\pi$ for every positively oriented simply closed path that enclosed the origin.

Solution:
Since C is arbitrary curve that encloses the origin, we need to consider some actually computable curve. Let C' be a counterclockwise circle center at 0 with radius $\varepsilon > 0$, where ε is small enough such that C' lies inside the interior of C. Let D be the region bounded by C and C'. Then its positively oriented boundary is given by $C \cup (-C')$ and so

$$\begin{aligned}\int_C M\,dx + N\,dy + \int_{-C'} M\,dx + N\,dy &= \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \\ &= \iint_D \left(\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2}\right) dA \\ &= 0\end{aligned}$$

hence $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C'} \mathbf{F}\cdot dr$. Then it follows from routine calculation that $\int_C \mathbf{F}\cdot dr = 2\pi$.

2 Stokes' Theorem

The *curl* of a vector field is given by

$$\operatorname{curl}\mathbf{F} := \nabla\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ M & N & P\end{vmatrix} = \left(\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z}\right)\mathbf{i}+\left(\frac{\partial M}{\partial z}-\frac{\partial P}{\partial x}\right)\mathbf{j}+\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mathbf{k}$$

Theorem 2.1 (Stokes' Theorem)**.** *Let S be a piecewise smooth oriented surface with a piecewise smooth boundary C. Let $\mathbf{F} = M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$ be a C^1 vector field on an open region containing S. Then*

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma$$

Exercise

Q2

Use the Stokes' theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, if $\mathbf{F} = xz\mathbf{i}+xy\mathbf{j}+3xz\mathbf{k}$ and C is the boundary of the portion of the plane $2x+y+z=2$ in the first octant, transversed counterclockwise as viewed from above.

Solution:
The unit normal vector is given by $\hat{\mathbf{n}} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{6}}(2,1,1)$.

$$\nabla\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xz & xy & 3xz\end{vmatrix} = (0, x-3z, y).$$

On S, we have

$$\nabla\times\mathbf{F} = (0, x-3(2-2x-y), y) = (0, 7x+3y-6, y)$$

then

$$(\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}} = \frac{1}{\sqrt{6}}(2,1,1)\cdot(0,7x+3y-6,y) = \frac{1}{\sqrt{6}}(7x+4y-6)$$

and

$$d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\mathbf{k}|}dA = \sqrt{6}\,dA$$

hence

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1\int_0^{2-2x}\frac{1}{\sqrt{6}}(7x+4y-6)\cdot\sqrt{6}\,dy\,dx = -1$$

3 Recordings