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Office Hour: Send me an email first, then we will arrange a meeting (if you need it).
Tutorial Arrangement:

- (1330 - 1355/ 15:30 - 15:55): Problems.
- (1355 - 1415/ 15:55 - 16:15): Class exercises.
- (1415 - 1430/ 16:15 - 16:30): Submission of Class Exercise via Gradescope.
- (1430 - 1530/ 16:30 - 17:30): Late submission period.

Announcement: There is a class exercise 9 but cancelled due to the reading week. It is now attached at the end of this tutorial notes, solution will be uploaded together with the solution of this tutorial notes later.

1 Green's Theorem

Theorem 1.1 (Green's Theorem). *let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a C^1-vector field in an open region containing R. Then*

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \oint_C M\, dx + N\, dy,$$

where C is oriented in the anticlockwise direction.

Q1

Let $\mathbf{F}(x, y) = (x - y)\mathbf{i} + x\mathbf{j}$ and R be a region bounded by the unit circle $C : \mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$ for $t \in [0, 2\pi]$. Verify that the Green's theorem holds

Solution:
Observe that $M(x, y) = x - y$ and $N(x, y) = x$. Then

$$\oint_C M\, dx + N\, dy = \oint_0^{2\pi} (\cos t - \sin t)(-\sin t\, dt) + \cos t(\cos t\, dt) = 2\pi$$

while

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_R (1 - (-1))\, dA = 2\pi(1)^2 = 2\pi.$$

Green's theorem holds for this question because $\mathbf{F}$ is clearly a smooth vector field, in particular C^1. Moreover, C is a unit circle, which is a smooth, simply closed curve enclosing the interior of the unit circle.

Q2

Evaluate the line integral

$$\oint_C (3y - e^{\sin x})\, dx + (7x + \sqrt{y^4 + 1})\, dy,$$

where C is the circle $x^2 + y^2 = 9$.

Solution:
The reason to learn Green's theorem is that when terms that are difficult to integrate appear, we can simply ignore them.

Check:

- $\mathbf{F}(x, y) = (3y - e^{\sin x})\mathbf{i} + (7x + \sqrt{y^4 + 1})\mathbf{j}$ is smooth.
  - You can check it by simply differentiating it. Since we only require C^1 here, you do not need to calculate every derivatives.
- C being the circle $x^2+y^2 = 9$ satisfies the condition stated in Green's theorem as discussed in Q1.

Then

$$\begin{aligned}\oint_C (3y - e^{\sin x})\, dx + (7x + \sqrt{y^4 + 1})\, dy &= \iint_R \frac{\partial}{\partial x}(7x + \sqrt{y^4 + 1}) - \frac{\partial}{\partial y}(3y - e^{\sin x})\, dA \\ &= \iint_R 7 - 3\, dA \\ &= 4\pi(3)^2 \\ &= 36\pi.\end{aligned}$$

As you can see, the terms $e^{\sin x}$ and $\sqrt{y^4 + 1}$ are gone.

2 Area

We can also find the area using Green's theorem.

Idea: Since $\iint_D 1\, dA$ gives the area of D, and the Green's theorem states

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_C M\, dx + N\, dy$$

therefore, if $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$, then the right hand side (after choosing appropriate M and N) yields the area of D. There are several possibilities:

1. $N(x, y) = x$, $M(x, y) = 0$
2. $N(x, y) = 0$, $M(x, y) = -y$
3. $N(x, y) = \frac{1}{2}x$, $M(x, y) = -\frac{1}{2}y$

Corollary 2.1 (Area formula)**.**

$$Area(D) = \oint_C x\, dy = -\oint_C y\, dx = \frac{1}{2}\oint_C x\, dy - y\, dx$$

Q3

Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution:
The parametrization of the ellipse is given by

$$\mathbf{r}(t) = (a\cos t, b\sin t).$$

Notice that

$$dx = -a\sin t\, dt, \quad dy = b\cos t\, dt$$

then

$$\begin{aligned}\frac{1}{2}\oint_C x\, dy - y\, dx &= \frac{1}{2}\int_0^{2\pi} a\cos t(b\cos t)\, dt - b\sin t(-a\sin t)\, dt \\ &= \pi ab.\end{aligned}$$

3 Class Exercise 9

No need to submit class exercise 9.

1. Find the area of the ellipse cut from the plane $z = cx$ by the cylinder $x^2 + y^2 = 1$.
2. Find the flux of $\mathbf{F} = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the plane $x = 0$, $x = 1$, and $z = 0$.

Solution:

(1) **Solution.** A parametrization is given by

$$\mathbf{s}(r,\theta) = (r\cos\theta, r\sin\theta, cr\cos\theta)\ , \quad (r,\theta) \in [0,1] \times [0,2\pi]\ .$$

We have $\mathbf{s}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} + c\cos\theta\mathbf{k}$ and $\mathbf{s}_\theta = -r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j} - cr\sin\theta\mathbf{k}$. Hence

$$\mathbf{s}_r \times \mathbf{s}_\theta = -cr\mathbf{i} + r\mathbf{k}\ .$$

The area of the ellipse is equal to

$$\iint_{D_1} |\mathbf{s}_r \times \mathbf{s}_\theta|\, dA(r,\theta) = \int_0^{2\pi}\int_0^1 \sqrt{1+c^2}\, r\, drd\theta = \sqrt{1+c^2}\,\pi\ .$$

(2) **Solution.** A parametrization of the surface is

$$\mathbf{r}(x,y) = (x, y, 4-y^2)\ , \quad (x,y) \in R \equiv [0,1] \times [-2,2]\ .$$

As $\mathbf{r}_x \times \mathbf{r}_y = -\varphi_x\mathbf{i} - \varphi_y\mathbf{j} + \mathbf{k} = 2y\mathbf{j} + \mathbf{k}$, the flux is equal to

$$\begin{aligned} & \iint_R \mathbf{F}(x,y,\varphi(x,y)) \cdot (2y\mathbf{j} + \mathbf{k})\, dA(x,y) \\ = & \iint_R (2xy - 3(4-y^2))\, dA(x,y) \\ = & \int_0^1 \int_{-2}^2 (2xy - 3(4-y^2))\, dydx \\ = & -32\ . \end{aligned}$$