Class Exercise 7

1. Evaluate
$$\int_C \frac{x^2}{y^{4/3}}\, ds,$$
where C is the curve $\gamma(t) = (t^2, t^3),\ t \in [1, 2]$.

Solution:
First we differentiate the curve: $\gamma'(t) = (2t, 3t^2)$, then the norm is given by
$$|\gamma'|(t) = \sqrt{(2t)^2 + (3t^2)^2} = \sqrt{4t^2 + 9t^4}.$$
Furthermore,
$$f(\gamma(t)) = \frac{(t^2)^2}{(t^3)^{4/3}} = 1.$$
Hence
$$\begin{aligned}\int_C \frac{x^2}{y^{4/3}}\, ds &= \int_1^2 \sqrt{4t^2 + 9t^4}\, dt \\ &= \int_1^2 t\sqrt{4 + 9t^2}\, dt \\ &= \frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right).\end{aligned}$$

2. Find the length of the arc of the parabola $y = x^2 - 3$ over $[0, 1]$.

Solution:
The idea of calculating the arc-length of a curve is to take the integrand of the line integral to be 1, i.e.,
$$L(\gamma) = \int_C ds.$$
Following this idea, we first parametrize $y = x^2 - 3$ over [0,1]. The easiest way is to let γ as follow:
$$\gamma(t) = (t, t^2 - 3) \quad \text{for } t \in [0, 1],$$
where the derivative and the corresponding norm are given by
$$\gamma'(t) = (1, 2t) \implies |\gamma'|(t) = \sqrt{1 + 4t^2}.$$
Hence
$$\begin{aligned}\int_C ds &= \int_0^1 \sqrt{1 + 4t^2}\, dt \\ &= \frac{1}{4}\left(2\sqrt{5} + \sinh^{-1}(2)\right).\end{aligned}$$

Remark: If you know how to calculate the arc-length of $y = f(x)$ over some interval $[a, b]$, then you can do so by directly applying it, i.e.,
$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_0^1 \sqrt{1 + (2x)^2}\, dx = \int_0^1 \sqrt{1 + 4x^2}\, dx$$
as you can see, they define the same integral.