Solution to Assignment 1

1. Consider the function $\varphi(x) = x^{-a}$ where a is positive for $x \in (0,1]$ and set $\varphi(0) = 1$ so that φ is a well-defined function on $[0,1]$. Show that φ is not integrable on $[0,1]$. This is the simplest example of an unbounded function.

   Solution. Assume on the contrary that φ is integrable on $[0,1]$ and let its integral be I. Given any number $\varepsilon > 0$, there is a partition P such that

   $$|\sum_j \varphi(x_j^*)\Delta x_j - I| < \varepsilon \ ,$$

   for any tags on P. (We don't care about the length of P.) Equivalently,

   $$-\varepsilon \le \sum_j \varphi(x_j^*)\Delta x_j - I \le \varepsilon \ .$$

   Taking $\varepsilon = 1$, say, we have

   $$\sum_j \varphi(x_j^*)\Delta x_j \le 1 + I \ .$$

   We dispose all summands in the summation above except the first summand to get

   $$\frac{1}{(x_1^*)^a}\Delta x_1 = \varphi(x_1^*)\Delta x_1 \le 1 + I \ .$$

   The right hand of this inequality is a finite number. However, if we choose the tag x_1^* very close to 0, the left hand side could be arbitrarily large, hence this inequality cannot be true. The contradiction shows that φ is not integrable.

   Note. Nonetheless, for $a \in (0,1)$ φ is improperly integrable. Will discuss it later.

2. Consider the function H in $\mathbb{R}^2$ defined by $H(x,y) = 1$ whenever x, y are rational numbers and equals to 0 otherwise. Show that H is not integrable in any rectangle.

   Solution. Let P be any partition of the rectangle. By choosing tags points (x^*, y^*) where x^* and y^* are rational numbers,

   $$\sum_{j,k} H(x_j^*, y_k^*)\Delta x_j \Delta y_k = \sum_{j,k} \Delta x_j \Delta y_k$$

   which is equal to the area of R. On the other hand, by choosing the tags so that x^* is irrational, $H(x^*, y^*) = 0$ so that

   $$\sum_{j,k} H(x_j^*, y_k^*)\Delta x_j \Delta y_k = \sum_{j,k} 0 \times \Delta x_j \Delta y_k = 0 \ .$$

   Depending the choice of tags, the Riemann sums are not the same for the same partition, hence they cannot tend to the same limit. We conclude that H is not integrable.

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Advanced Calculus II
Assignment 1 Solution

Q20

$$\iint_R xye^{xy^2}\,dA, \qquad R:\ 0 \le x \le 2,\ 0 \le y \le 1$$

Solution:

$$\begin{aligned}
\int_0^2 \int_0^1 xye^{xy^2}\,dy\,dx &= \int_0^2 \frac{1}{2}e^{xy^2}\Big|_0^1\,dx \\
&= \int_0^2 \frac{1}{2}(e^x - 1)\,dx \\
&= \frac{1}{2}\left[e^x - x\right]_0^2 \\
&= \frac{1}{2}(e^2 - 3)
\end{aligned}$$

Q32

Evaluate

$$\int_{-1}^1 \int_0^{\pi/2} x\sin\sqrt{y}\,dy\,dx$$

Solution:

$$\begin{aligned}
\int_{-1}^1 \int_0^{\pi/2} x\sin\sqrt{y}\,dy\,dx &= \int_0^{\pi/2} \int_{-1}^1 x\sin\sqrt{y}\,dx\,dy \\
&= 0
\end{aligned}$$

because x is an odd function.

Q34

Use Fubini's theorem to evaluate

$$\int_0^1 \int_0^3 xe^{xy}\,dx\,dy$$

Solution:

$$\begin{aligned}
\int_0^1 \int_0^3 xe^{xy}\,dx\,dy &= \int_0^3 \int_0^1 xe^{xy}\,dy\,dx \\
&= \int_0^3 e^{xy}\big|_0^1\,dx \\
&= \int_0^3 e^x - 1\,dx \\
&= e^3 - 4
\end{aligned}$$