Solution to Class Exercise 5

1. Determine the mass and the center of mass of the thin solid region bounded in the first quadrant bounded by the coordinate axes and the line $x+2y=1$. The density of the solid is $\delta(x,y)=x$.

   Solution. The mass of the region is
$$\iint_D x\,dA = \int_0^1\int_0^{(1-x)/2} x\,dydx = \frac{1}{12}\ .$$
   Next,
$$M_y = \iint_D x^2\,dydx = \int_0^1\int_0^{(1-x)/2} x^2\,dydx = \frac{1}{24}\ .$$
   Also,
$$M_x = \iint_D yx\,dA = \int_0^1\int_0^{(1-x)/2} xy\,dydx = \frac{1}{96}\ .$$
   The center of mass of this region is
$$(\bar{x},\bar{y}) = \frac{1}{M}(M_y, M_x) = \left(\frac{1}{2},\frac{1}{8}\right)\ .$$

2. Let Ω be the region bounded between the surface $z=9-x^2-y^2$ and $z=5$. Express
$$\iiint_\Omega f(x,y,z)\,dV$$
   in cylindrical and spherical coordinates.

   Solution. These two surfaces intersect at $(x,y,5)$ where (x,y) belongs to the circle $x^2+y^2=4$. In cylindrical coordinates,
$$\iiint_\Omega f\,dV = \int_0^{2\pi}\int_0^2\int_5^{9-r^2} f(r\cos\theta, r\sin\theta, z)r\,dzdrd\theta\ .$$
   Any ray of angle $\varphi\in[0,\varphi_0], \varphi_0=\tan^{-1}2/5$, hits $z=5$ first and then $z=9-x^2-y^2$. In spherical coordinates,
$$\iiint_\Omega f\,dV = \int_0^{2\pi}\int_0^{\varphi_0}\int_{5/\cos\varphi}^{\rho_0(\varphi)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi\,d\rho d\varphi d\theta\ .$$
   where $\rho_0(\varphi)$ is the positive solution of $\rho\cos\varphi = 9-\rho^2\sin^2\varphi$ for each fixed $\varphi\in[0,\varphi_0]$, i.e.
$$\rho_0(\varphi) = \frac{-\cos\varphi+\sqrt{\cos^2\varphi+36\sin^2\varphi}}{2\sin^2\varphi}$$

3. The same problem as in (2) where Ω is replaced by H, the region bounded by $z=9-x^2-y^2, z=5$ and $z=0$.

   Solution. Need to consider the region over the disk $x^2+y^2\le 4$ and over the annulus $4\le x^2+y^2\le 9$ separately. In cylindrical coordinates,
$$\iiint_H f\,dV = \int_0^{2\pi}\int_2^3\int_0^{9-r^2} f(r\cos\theta, r\sin\theta, z)r\,dzdrd\theta + \int_0^{2\pi}\int_0^2\int_0^5 f(r\cos\theta, r\sin\theta, z)r\,dzdrd\theta\ .$$
   In spherical coordinates,
$$\iiint_H f\,dV = \int_0^{2\pi}\int_{\varphi_0}^{\pi/2}\int_0^{\rho_0(\varphi)} f(\cdot,\cdot,\cdot)\rho^2\sin\varphi\,d\rho d\varphi d\theta + \int_0^{2\pi}\int_0^{\varphi_0}\int_0^{5/\cos\varphi} f(\cdot,\cdot,\cdot)\rho^2\sin\varphi\,d\rho d\varphi d\theta\ .$$