

Solution to Assignment 3

Supplementary Problems

1. Express the straight line $ax + by = 1$, $a, b > 0$, in polar coordinates. How about $ax + by = 0$?

Solution. Let $c = \sqrt{a^2 + b^2}$. Equation is

$$1 = r(a \cos \theta + b \sin \theta) = rc \left(\frac{a}{c} \cos \theta + \frac{b}{c} \sin \theta \right) = rc \sin(\theta + \alpha),$$

where α satisfies $\sin \alpha = a/c$. In polar coordinates, the straight line is given by

$$r = \frac{1}{c \sin(\theta + \alpha)}, \quad \theta \in (-\alpha, -\alpha + \pi).$$

2. Express the hyperbola $x^2 - y^2 = 1$ ($y \geq 0$) in polar coordinates.

Solution. From $1 = r^2(\cos^2 \theta - \sin^2 \theta) = r^2 \cos 2\theta$ we get

$$r = \frac{1}{\sqrt{\cos 2\theta}},$$

where $\theta \in (-\pi/4, \pi/4)$.

3. Discuss the existence of the improper integral

$$\iint_D \frac{y}{(x^2 + y^2)^{3/2}},$$

where D is the region enclosed by the polar graph $r = 1 + \cos \theta$.

Solution. Let a be a small positive number. We consider

$$I(a) \equiv \iint_{D_a} \frac{y}{(x^2 + y^2)^{3/2}},$$

where D_a is the region bounded between $r = 1 + \cos \theta$, $y \geq 0$ and $r = a$. Using polar coordinates,

$$I(a) = \int_0^{\theta_0} \int_a^{1+\cos \theta} \frac{r \sin \theta}{r^3} r \, dr \, d\theta,$$

where θ_0 satisfies $1 + \cos \theta_0 = a$. Hence

$$I(a) = \int_0^{\theta_0} \sin \theta (\log(1 + \cos \theta) - \log a) \, d\theta.$$

On one hand, we have

$$\begin{aligned} \int_0^{\theta_0} \sin \theta \log(1 + \cos \theta) \, d\theta &= - \int_1^{\cos \theta_0} \log(1 + t) \, dt \\ &= -(1 + \cos \theta_0) \log(1 + \cos \theta_0) + \cos \theta_0 + 2 \log 2 - 1 \\ &\rightarrow 2 \log 2 - 2. \end{aligned}$$

as $a \rightarrow 0$ ($\theta_0 \rightarrow \pi$). On the other hand,

$$- \int_0^{\theta_0} \sin \theta \log a \, d\theta = \log a (\cos \theta_0 - 1) \rightarrow -\infty$$

as $a \rightarrow 0$. That is,

$$\lim_{a \rightarrow 0} I(a) = -\infty,$$

the improper integral does not exist.

MATH2020B HW3 Solution.

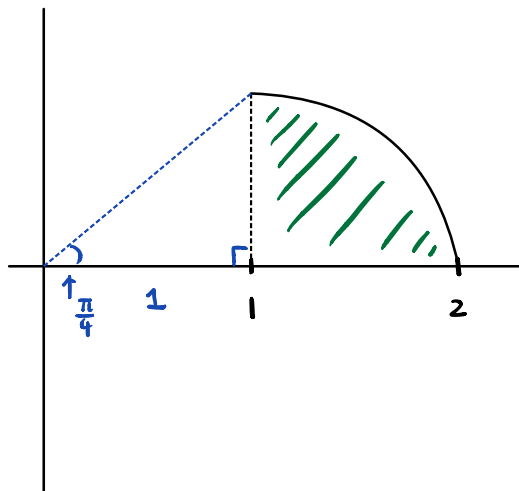
Q22:

$$\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2} dy dx$$

$$= \int_0^{\pi/4} \int_{\frac{1}{\cos\theta}}^{2\cos\theta} \frac{1}{r^4} r dr d\theta$$

$$= (\text{Steps ...})$$

$$= \frac{\pi}{16}$$

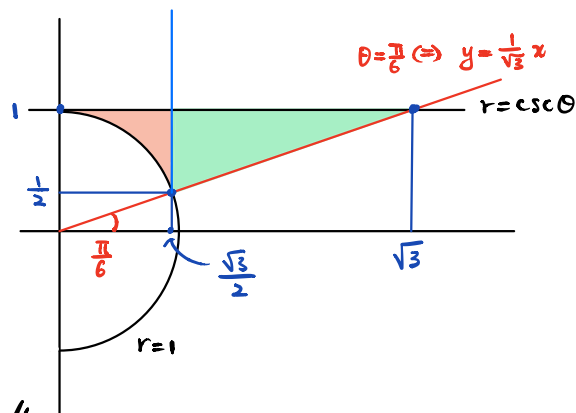


- ① $2x - x^2 = y^2$
 $x^2 + y^2 - 2x = 0 \Rightarrow r^2 - 2r\cos\theta = 0$
 $\Rightarrow (x-1)^2 + y^2 = 1$
 $\Rightarrow r = 2\cos\theta$
- ② $0 \leq \theta \leq \frac{\pi}{4}, \frac{1}{\cos\theta} \leq r \leq 2\cos\theta$

Q24

$$\int_{\pi/6}^{\pi/2} \int_1^{\csc\theta} r^2 \cos\theta dr d\theta$$

$$\left[\begin{array}{l} r^2 \cos\theta dr d\theta \\ = (r \cos\theta) r dr d\theta \end{array} \right]$$



$$\text{[Diagram of two triangles: a small orange one and a larger green one]} = \int_0^{\sqrt{3}/2} \int_{\sqrt{1-x^2}}^1 x dy dx + \int_{\sqrt{3}/2}^{\sqrt{3}} \int_{\frac{x}{\sqrt{3}}}^1 x dy dx$$

